Pascal's triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Rows zero to five of Pascal's triangle

In mathematics, **Pascal's triangle** is a triangular array of the binomial coefficients. In the Western world, it is named after French mathematician Blaise Pascal, although other mathematicians studied it centuries before him in India,[1] Persia (Iran), China, Germany, and Italy.

$\binom{n}{n} = 1$

The rows of Pascal's triangle (sequence A007318 in OEIS) are conventionally enumerated starting with row *n* = 0 at the top (the 0th row). The entries in each row are numbered from the left beginning with *k* = 0 and are usually staggered relative to the numbers in the adjacent rows. Having the indices of both rows and columns start at zero makes it possible to state that the binomial coefficient appears in the *n*th row and *k*th column of Pascal's triangle. A simple construction of the triangle proceeds in the following manner: In row 0, the topmost row, the entry is (the entry is in the zeroth row and zeroth column). Then, to construct the elements of the following rows, add the number above and to the left with the number above and to the right of a given position to find the new value to place in that position. If either the number to the right or left is not

present, substitute a zero in its place. For example, the initial number in the first (or any other) row is 1 (the sum of 0 and 1), whereas the numbers 1 and 3 in the third row are added to produce the number 4 in the fourth row.
This construction is related to the binomial coefficients by Pascal's rule, which says that if

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

for any non-negative integer *n* and any integer *k* between 0 and *n*.
Pascal's triangle has higher dimensional generalizations. The three-dimensional version is called *Pascal's pyramid* or *Pascal's tetrahedron*, while the general versions are called *Pascal's simplices*.

Binomial expansions

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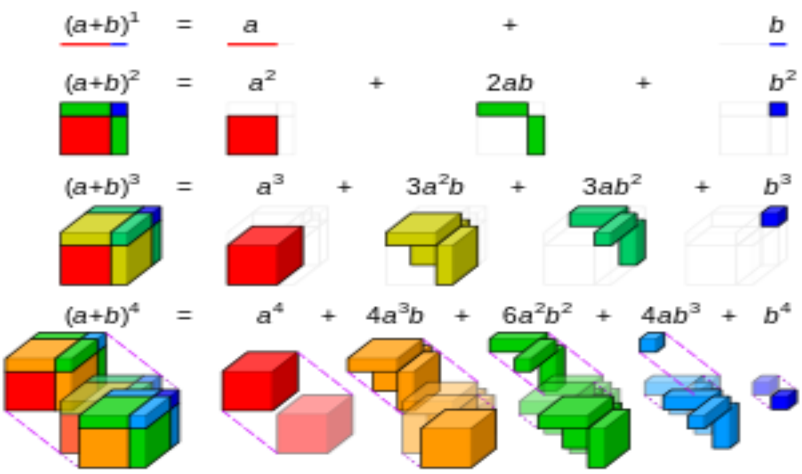

Visualisation of binomial expansion up to the 4th power

Pascal's triangle determines the coefficients which arise in binomial expansions. For an example, consider the expansion

$(x + y)^2 = x^2 + 2xy + y^2 = \mathbf{1}x^2y^0 + \mathbf{2}x^1y^1 + \mathbf{1}x^0y^2$.

Notice the coefficients are the numbers in row two of Pascal's triangle: 1, 2, 1. In general, when a binomial like $x + y$ is raised to a positive integer power we have:

$(x + y)^n = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_{n-1}xy^{n-1} + a_ny^n$,

where the coefficients a_i in this expansion are precisely the numbers on row n of Pascal's triangle. In other words,

$$a_i = \binom{n}{i}.$$

This is the binomial theorem.

Notice that the entire right diagonal of Pascal's triangle corresponds to the coefficient of y^n in these binomial expansions, while the next diagonal corresponds to the coefficient of xy^{n-1} and so on.

To see how the binomial theorem relates to the simple construction of Pascal's triangle, consider the problem of calculating the coefficients of the expansion of $(x + 1)^{n+1}$ in terms of the corresponding coefficients of $(x + 1)^n$ (setting $y = 1$ for simplicity). Suppose then that

$$(x+1)^n = \sum_{i=0}^{n} a_i x^i.$$