Compounding Interest and e ,Proof of $\int (1/x)dx = Loge(x)$

Date: 11/11/98 at 11:52:55
From: John Radway
Subject: How in the world did they come up with e?

I know that e is equal to 2.718281828..., and that it is the base of natural logarithms, but what does that number mean? How is it calculated? How did they come up with it?

Date: 11/11/98 at 14:11:24
From: Doctor Anthony
Subject: Re: How in the world did they come up with e?

In the 1730s Euler investigated the result of compounding interest continuously when a sum of money, say, is invested at compound interest.

If interest is added once a year we have the usual formula for the amount, A, with principal P, rate of interest r percent per annum, and t the time in years:

A = P(1 + r/100)^t

If interest is added twice a year, we replace r by r/2 and t by 2t. So the formula becomes:

A = P(1 + r/(2 * 100))^(2t) = amount after t years.

If 3 times a year then A at the end of t years would be:

A = P(1 + r/(3 * 100))^(3t)

and if we added interest N times a year, then after t years the amount
A would be:

A = P[1 + r/(N * 100)]^(Nt)

Now to simplify the working, we put r/(100N) = 1/n so N = nr/100 and:

$$A = P[1 + 1/n]^{(nrt/100)}$$
$$A = P[(1 + 1/n)^n]^{(rt/100)}$$

We now let n -> infinity and we must see what happens to the expression
$(1 + 1/n)^n$ as n tends to infinity.

Expanding by the binomial theorem:

$$(1 + 1/n)^n = 1 + n(1/n) + n(n-1)/2!\,(1/n)^2 + n(n-1)(n-2)/3!\,(1/n)^3 + \ldots$$

Now take the n's in the denominators of $1/n^2$, $1/n^3$, and so on, and
distribute one n to each of the terms n, n-1, n-2, etc in the numerator, getting for each arbitrary term of the sequence:

$$1 * (1-1/n) * (1-2/n) * \ldots$$

So we now have:

$$(1 + 1/n)^n = 1 + 1 + 1(1-1/n)/2! + 1(1-1/n)(1-2/n)/3! + \ldots$$

Now let n -> infinity and the terms 1/n, 2/n, etc. all go to zero, giving:

$$(1 + 1/n)^n = 1 + 1 + 1/2! + 1/3! + 1/4! + \ldots..$$

and this series converges to the value we now know as e.

If you consider e^x you get:

$$(1 + 1/n)^{(nx)}$$

and expanding this by the binomial theorem you have:

$$(1 + 1/n)^{(nx)} = 1 + (nx)(1/n) + nx(nx-1)/2!\,(1/n)^2 + \ldots.$$

and carrying through the same process of putting the n's in the

denominator into each term in the numerator as described above you
obtain:

$$e^x = 1 + x + x^2/2! + x^3/3! + ...$$

and differentiating this we get:

$$d(e^x)/dx = 0 + 1 + 2x/2! + 3x^2/3! + ...$$
$$= 1 + x + x^2/2! + x^3/3! + ...$$
$$= e^x$$

Reverting to our original problem of compounding interest continuously,
the formula for the amount becomes:

$$A = P e^{(rt/100)}$$

You might like to compare the difference between this and compounding
annually.

If P = 5000, r = 8, t = 12 years:

Annual compounding gives $A = 5000(1.08)^{12} = 12590.85$
Continuous compounding gives $A = 5000 e^{(96/100)} = 13058.48$
The difference is not as great as might be expected.

- Doctor Anthony, The Math Forumhttp://mathforum.org/dr.math/

Proof that INT(1/x)dx = lnx

Date: 11/08/96 at 22:14:34
From: Ben Faulkner
Subject: Integrating (1/x)dx

My friend and I would like to know how to integrate (1/x)dx.
Our teacher just told us that we couldn't, or at least not yet.
Is there a way?

Date: 11/09/96 at 08:17:59

From: Doctor Anthony
Subject: Re: Integrating (1/x)dx

Dear Ben and Friend,

The integral of 1/x is intimately bound up with the exponential and log functions. Some people like to start with the log function and get an expression for the exponential function, or, and I prefer this method, begin with exponential functions and then move to the log function.

Exponential growth can be thought of as compound interest growth, with
the intervals of adding interest getting shorter and shorter until it is continuous growth - as would be the case for growth of a plant or animal.

The formula for compound interest is $A = P(1 + r/100)^t$ where A = amount, P = principal, r = rate percent per annum and t = time in years.

If interest were added twice yearly, this formula would become:

$$A = P(1 + r/200)^{(2t)}$$

If interest were added s times a year:

$$A = P(1 + r/(100s))^{st}$$

Now put $r/(100s) = 1/n$, so $s = nr/100$:

$$A = P(1 + 1/n)^{(nrt/100)}$$

$$A = P[(1 + 1/n)^n]^{(rt/100)}$$

Now if we let $n \to \infty$ so that interest is added continuously, the expression in [ ] brackets $(1 + 1/n)^n$ is denoted by e (by definition), and we have:

$$A = Pe^{(rt/100)}$$

We can evaluate e using the binomial theorem. If you want to learn more about the binomial theorem, take a look at:

http://mathforum.org/dr.math/problems/fama.7.10.96.html

Continuing on with evaluating $(1 + 1/n)^n$:

$(1 + 1/n)^n = 1 + n(1/n) + (n(n-1)/2!)(1/n^2) + (n(n-1)(n-2)/3!)(1/n^3) + ..$

Dividing the n's on the bottom line into the brackets we get:

$= 1 + 1 + 1(1-1/n)/2! + 1(1-1/n)(1-2/n)/3! + \ldots$

As $n \to$ infinity:

$= 1 + 1 + 1/2! + 1/3! + \ldots$.. to infinity

This series gives $e = 2.71828183$ to 8 places, but e in decimal form is an unending, non-repeating, irrational number like pi or sqrt(3).

If you now consider $(1 + 1/n)^{(nx)} = e^x$ and expand again by the binomial theorem you get:

$e^x = 1 + x + x^2/2! + x^3/3! + \ldots$ to infinity.

Now comes the very important part. If you differentiate e^x you get

$d(e^x)/dx = 0 + 1 + 2x/2! + 3x^2/3! +$ etc

$= 1 + x + x^2/2! +$ etc

$= e^x$ (back to where we started)

So the function e^x has the UNIQUE property that it is equal to its differential coefficient.

So if $y = e^x$ then $dy/dx = e^x = y$ $dy/dx = y$

Then $dy/y = dx$ and integrating

INT[dy/y] = x + const.

So to solve this integral dy/y in terms of y we need to express x in terms of y.

We had e^x = y, so we take the natural logarithm (which means logs to
base e and is denoted by "ln") of each side and we have:

x = ln(y)

Now we see how to integrate dy/y. We have

INT[dy/y] = ln(y) + const.

This would be the expression whatever letter is the variable. So if x is the variable we can write:

INT[dx/x] = ln(x) + const.

As you can see it is quite a long story, but it is a watershed in advancing to new branches of mathematics, with greatly improved applications to real world problems.

-Doctor Anthony, The Math Forum
Check out our web site! http://mathforum.org/dr.math/