英語で 数学・物理とモデリングを学ぶ。

文献1. 英数字・数式の読み方；上智大学理工学部

文献2.The Math　Forum　ー　Ask Dr.Math

英語で数学をビデオ学習できます。その際には、windowsでもiTunesを下記よりdownloadし、

http://www.apple.com/jp/itunes/download/

iTunes　Uサイトに集約された。著名大学のコンテンツで学習することをお勧めします。

1）iTunes Uサイト：Math103、Collage　Algebra　-　PC

2) iTunes Uサイト：Math110、Applied Calculus for Business – PC

講義の概要：前期と後期の2回に分けている。大学1年生向け。

目標１：2次方程式を因数分解でとく。

目標２：2項展開とそのなかのフィボナッチ数を発見する。

目標3：フィボナッチ数を生む漸化式もとめ、2次多項式（特性多項式）の解から
　　　黄金数を求める。

目標4：黄金比と5角形、螺旋の関係を理解する。

目標5：複素数の定義と計算を複素平面上で理解する。例、ペンタゴンの黄金数の計算

目標６：複素平面上の正多角形を生み出すx^n=1の円分多項式を理解。併せて、三角関数、ドモアブルの定理などを理解する。ニュートンの計算など

目標７：微分の理解。2項展開を使って、X^nの微分を定義からの極限として理解する。

目標８：極限の計算。eの意味を、２項展開、複利計算の極限として理解する。指数関数の微分がなぜe^xかを示す。

目標８：微分を使って、表面積一定で容積最大となるアイスクリームのコーンの形状などを、解析的に求めてみる。（数値計算は後期）

目標9：関数を多項式近似するために、マクローリン展開を導出し理解する。

　指数関数をマクローリン展開し、sin(x)、cos(x)の展開式を求めてみる。

目標10:フィボナッチ数列などの線形漸化式の性質を知る。

目標11：線形漸化式で表現できる数学的モデルとして、１.タンクと２．単振動を理解する。

目標12:フックの法則から2階微分式を与え、オイラー差分で近似して、積分を行ってみる。

　　　　このことから差分を使った漸化式から積分計算できることを習う。オイラー法。

目標13：重力モデルから重力圏を計算し脱出速度を求めてみる。

目標14:ニュートン法をの式をマクローリン展開から求める。これを使って、解析的に解が求められない場合でも、数値計算で解を求められることを理解する。

前期

第1回　英語で数式を読む。2次方程式を解く。多項式の根。2項展開（x+y)^n　2項係数とパスカルの三角形。nCi の和 = 2^n

第2回　パスカルの三角形のなかのフィボナッチ数列。数列と漸化式。フィボナッチ数列の一般解。（線形漸化式の解と性質）

第3回　黄金分割と黄金比、ケプラーの三角形。等角らせんのフィボナッチ近似。

第4回　多項式とfactoring。f((x)=x^n　-1　=0　を解く。ｎ個の解を複素平面上に図示。

　　　　　複素数の計算、複素方程式。

第5回 ガウス空間、極座標表示、円分多項式、ドモアブルの定理。

　　　　　複素平面上の正多角形の計算。黄金比を与える正5角形。

第6回 極限とは。2項定理から、ネイピア数ｅ　を求める。連続複利

計算とe^x

　　　　指数関数y=e^xの微分は何故e^xか。　オイラーの公式と等式（e＾iπ + 1 = 0）。

第7回 極限と微分。解析解を求める。

　　　　表面積一定で容積最大となるアイスクリームのコーンの形状は？

　　　　容積一定で表面積を最大とする缶の形状は？。

第8回　方程式の多項式近似（マクローリン展開、テーラー展開）。sin(Θ)を求める。

第8回　微分とは、何故d(x^n)/dx=nx^(n-1)か。微分計算の復習。カセットテープの回転運動。

第9回 ばねと単振動。フックの法則、運動の表現。2次の線形微分方程式。

　　　　オイラー差分近似による位置と速度の漸化式。漸化式を繰り返すことで、

　　　　積分値（単振動）が求められる。

第10回 差分方程式の数値計算。運動行程式の解をオイラー法で求める。

第11回 重力、引力。ニュートンの運動方程式、オイラーの運動方程式

第12回 差分・微分方程式。初期値と解の数列。線形性とは何か？。重ね合わせの原理。

第13回　最適化計算の事例（ニュートン法、点列を近似する曲線など）

第14回 吊り橋や構造物の懸垂曲線やアーチ。その導出と微分方程式は？

第15回 ラグランジュの運動方程式

YouTube　で学習：Khan Academy　でリスニング

http://www.khanacademy.org/
米国で最近話題のオンラインの教育動画サイト　Khan　Academy。
TOEFLやTOEICのリスニングで扱うような学術的内容が多い。

代数：Ａlgebra　http://www.khanacademy.org/#algebra

1. Imaginary Roots of Negative Numbers
2. Complex Conjugates Example
3. Adding Complex Numbers
4. Subtracting Complex Numbers
5. Multiplying Complex Numbers
6. Dividing Complex Numbers
7. Complex Roots from the Quadratic Formula

解析学：Calculus　http://www.khanacademy.org/#calculus

1. Introduction to Limits
2. Limit Examples (part 1)
3. Limit Examples (part 2)
4. Proof: lim (sin x)/x
5. Proof: d/dx(x^n)
6. Proof: d/dx(sqrt(x))
7. Proof: d/dx(ln x) = 1/x
8. Proof: d/dx(e^x) = e^x
9. Maclauren and Taylor Series Intuition
10. Cosine Taylor Series at 0 (Maclaurin)
11. Sine Taylor Series at 0 (Maclaurin)
12. Taylor Series at 0 (Maclaurin) for e to the x

13. Euler's Formula and Euler's Identity
14. Visualizing Taylor Series Approximations
15. Generalized Taylor Series Approximation
16. Visualizing Taylor Series for e^x
17. Taylor Polynomials
18. Exponential Growth

物理のための解析学　http://www.khanacademy.org/#Precalculus

1. Combinations
2. Binomial Theorem (part 1)
3. Binomial Theorem (part 2)
4. Binomial Theorem (part 3)
5. Introduction to interest
6. Interest (part 2)
7. Introduction to compound interest and e
8. Compound Interest and e (part 2)
9. Compound Interest and e (part 3)
10. Compound Interest and e (part 4)
11. Exponential Growth
12. Basic Complex Analysis
13. Exponential form to find complex roots
14. Complex Conjugates
15. Series Sum Example
16. Complex Determinant Example
17. Introduction to interest
18. Interest (part 2)

物理のための三角法　http://www.khanacademy.org/#Trigonometry

1. Radians and degrees
2. Trigonometric Identities
3. Proof: sin(a+b) = (cos a)(sin b) + (sin a)(cos b)

4. Proof: cos(a+b) = (cos a)(cos b)-(sin a)(sin b)
5. Trig identities part 2 (parr 4 if you watch the proofs)
6. Trig identies part 3 (part 5 if you watch the proofs)

Solving Quadratic Equations
I am very confused on how to do these problems.
Please help me.

n次多項式の根と虚数：代数の基本定理

An n^{th} degree equation can be written in modern notation as

$$x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n = 0$$

where the coefficients $a_1, \ldots, a_{n-2}, a_{n-1}$, and a_n are all constants. Girard said that an n^{th} degree equation admits of n solutions, if you allow all roots and count roots with multiplicity. So, for example, the equation $x^2 + 1 = 0$ has the two solutions $\sqrt{-1}$ and $-\sqrt{-1}$, and the equation $x^2 - 2x + 1 = 0$ has the two solutions 1 and 1. Girard wasn't particularly clear what form his solutions were to have, just that there be n of them: $x_1, x_2, \ldots, x_{n-1}$, and x_n.

Girard gave the relation between the n roots $x_1, x_2, \ldots, x_n$, and x_n and the n coefficients $a_1, \ldots, a_{n-2}, a_{n-1}$, and a_n . First, the sum of the roots $x_1 + x_2 + \ldots, + x_n$ is $-a_1$, the negation of the coefficient of x^{n-1} . Next, the sum of all products of pairs of solutions is a_2. Next, the sum of all products of triples of solutions is $-a_3$. And so on until the product of all n solutions is either a_n (when n is even) or $-a_n$ (when n is odd).

Here's an example. The 4^{th} degree equation

$$x^4 - 6x^3 + 3x^2 + 26x - 24 = 0$$

has the four solutions –2, 1, 3, and 4. The sum of the solutions equals 6, that is –2 + 1 + 3 + 4 = 6. The sum of all products of pairs (six of them) is

$$(-2)(1) + (-2)(3) + (-2)(4) + (1)(3) + (1)(4) + (3)(4)$$

which is 3. The sum of all products of triples (four of them) is

$$(-2)(1)(3) + (-2)(1)(4) + (-2)(3)(4) + (1)(3)(4)$$

which is 26. And the product of all four solutions is –24.

Descartes (1596–1650) also studied this relation between solutions and coefficients, and showed more explicitly why the relationship holds. Descartes called negative solutions "false" and treated other solutions (that is, complex numbers) "imaginary".

Binomial Expansions and Pascal's Triangle

Can you supply the definition of what a binomial expansion is, where it would be used, why, and how to do one? This would be a great help
because I may be able to use it for forecasting.

Subject: Re: Binomial Expansions

A *binomial* is a polynomial expression with two terms, like x+y, x^2+1 (x squared plus 1), or x^4-3*x.

Binomial expansion refers to a formula by which one can "expand out"
expressions like (x+y)^5 and (3*x+2)^n, where the entire binomial is raised to some power. Usually, binomial expansion is introduced using
a construction called Pascal's Triangle, but I prefer to think of it in terms of something called the *binomial coefficient*, which I'll explain later.

First, we'll look at the "generic" binomial x+y, and its powers (x+y)^2, (x+y)^3, ... (x+y)^n. Notice the following:

(x+y)^1 = x+y
(x+y)^2 = (x+y)(x+y) = x^2+2*x*y+y^2
(x+y)^3 = (x+y)(x+y)^2 = x^3+3*x^2*y+3*x*y^2+y^3
(x+y)^4 = (x+y)(x+y)^3 = x^4+4*x^3*y+6*x^2*y^2+4*x*y^3+y^4
...

$$\binom{n}{k} = {}_n\mathrm{C}_k = \frac{n!}{(n-k)!\,k!}.$$

パスカルの三角形

2 項係数は次のようにしても求めることができる。

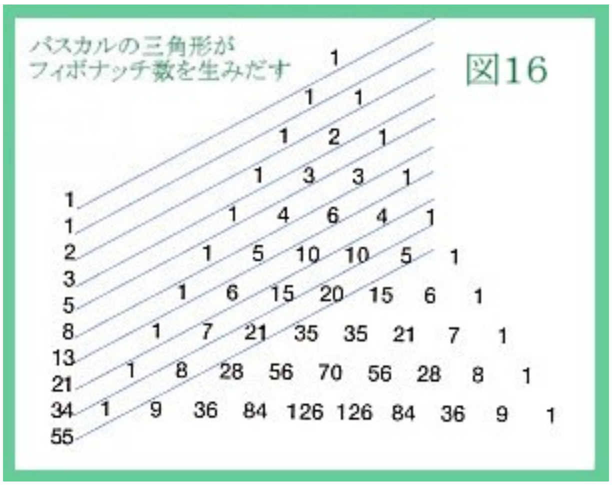
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

この関係式はしばしば n 段目の k 番目に ${}_n\mathrm{C}_k$ を配置（もちろん n も k も 0 から数え始める）した三角形として表され、パスカルに因んでパスカルの三角形という。

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1 \\
1 \quad 10 \quad 45 \quad 120 \quad 210 \quad 252 \quad 210 \quad 120 \quad 45 \quad 10 \quad 1
\end{array}$$

ある次数の 2 項係数は、左上と右上にある前の次数の 2 項係数 2 つを足したものになる。数が書いていない空白は 0 と考える。

パスカルの三角形のなかのフィボナッチ数



パスカルの三角形から二項係数が求まることは、分配法則と数学的帰納法を用いれば明らかである。実際、

$$(x+y)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k-1}$$

と表されたならば、この両辺に $x + y$ を掛けることにより

$$(x+y)^n = x^n + y^n + \sum_{k=1}^{n-1} \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right) x^k y^{n-k}$$

が成立することが確かめられる。これを

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

と比較すれば係数について所期の関係を得る。

What is Pascal's Triangle?
How do you construct it?
What is it used for?

Pascal's Triangle is an arithmetical triangle you can use for some neat things in mathematics. Here's how you construct it:
1
1 1
1 2 1

1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
.
.
.
You start out with the top two rows: 1, and 1 1. Then to construct each entry in the next row, you look at the two entries above it (i.e. the one above it and to the right, and the one above it and to the left). At the beginning and the end of each row, when there's only one number above, put a 1. You might even think of this rule (for placing the 1's) as included in the first rule: for instance, to get the first 1 in any line, you add up the number above and to the left (since there is no number there, pretend it's zero) and the number above and to the right (1), and get a sum of 1.
When people talk about an entry in Pascal's Triangle, they usually give a row number and a place in that row, beginning with row zero and place zero. For instance, the number 20 appears in row 6, place 3.

That's how you construct Pascal's Triangle. Here's an interactive version where you can specify the number of rows you want to see and from which you can bring up a large version that goes through row 19.

Where do we use Pascal's Triangle?
Pascal's Triangle is more than just a big triangle of numbers. There are two major areas where Pascal's Triangle is used, in Algebra and in Probability / Combinatorics.

Algebra

Let's say you have the polynomial x+1, and you want to raise it to some powers, like 1,2,3,4,5,.... If you make a chart of what you get when you do these power-raisings, you'll get something like this:

(x+1)^0 = 1
(x+1)^1 = 1 + x
(x+1)^2 = 1 + 2x + x^2
(x+1)^3 = 1 + 3x + 3x^2 + x^3
(x+1)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4
(x+1)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 .....

If you just look at the coefficients of the polynomials that you get, you'll see Pascal's Triangle! Because of this connection, the entries in Pascal's Triangle are called the binomial coefficients.
There's a pretty simple formula for figuring out the binomial coefficients:

n!
[n:k] = --------
k! (n-k)!
6 * 5 * 4 * 3 * 2 * 1
For example, [6:3] = ------------------------ = 20.
3 * 2 * 1 * 3 * 2 * 1
Probability/Combinatorics
The other main area where Pascal's Triangle shows up is in Probability, where it can be used to find Combinations. Let's say you have five hats on a rack, and you want to know how many different ways you can pick two of them and wear them. It doesn't matter to you which hat is on top, it just matters which two hats you pick. So this problem amounts to the question "how many different ways can you pick two objects from a set of five objects?"
The answer? It's the number in the second place in the fifth row, i.e. 10. (Remember that the first number in the row, 1, is always place 0.)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

1 7 21 35 35 21 7 1
Because of this choosing property, the binomial coefficient [6:3] is usually read "six choose three." If you want to find out the probability of choosing one particular combination of two hats, then that probability is 1/10.
In about 1654 Blaise Pascal started to investigate the chances of getting different values for rolls of the dice, and his discussions with Pierre de Fermat are usually considered to have laid the foundation for the theory of probability.

Triangular Numbers, Fibonacci Numbers
The triangular numbers and the Fibonacci numbers can be found in Pascal's triangle. The triangular numbers are easier to find: starting with the third one on the left side go down to your right and you get 1, 3, 6, 10, etc.
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

The Fibonacci numbers are harder to locate. To find them you need to go up at an angle: you're looking for 1, 1, 1+1, 1+2, 1+3+1, 1+4+3, 1+5+6+1.
Prove Sum nCi = 2^n ...

I would like to know how to prove sum nCi = 2^n with 0 <= i <= n.
I tried the definition of nCi = n!/[i!*(n-i)!] but nothing.

Thanks,

Subject: Re: Pascal's triangle

Thanks for writing to us, Andy! This is a good one.

It comes from the use of the nCi in the binomial theorem:

[1 + x]^n = 1 + nC1*x + nC2*x^2 + nC3*x^3 + ... + nC(n-1)*x^(n-1) + x^n.

Since nC0 = 1, and nCn = 1 also, the coefficients in the formula above
are just the terms in your sum. Now if you let x = 1, the left side becomes 2^n and the right side is nC0 + nC1 + nC2 + ... + nCn, which is
what you want!

But that's of no use unless you already knew the binomial theorem. So to
prove it from scratch, you first have to show that

nCk + nC(k-1) = (n+1)Ck,

which is easy using "nCi = n!/[i!*(n-i)!]" and a little algebra.

Then you prove the main result using the Principle of Mathematical Induction. The method is like this:

Suppose you want to prove that some formula A(n) = B(n) for all integers n.

You first show that it is true when n = 1. In other words, you show the special case A(1) = B(1). [This is sometimes called the priming step.]

Now you argue as follows. Suppose it is always true that whenever A(k) = B(k). Then it follows that A(k+1) = B(k+1). This is a lot easier to show than to show that A(n) = B(n). Okay, suppose you can
show this. Then starting from A(1) = B(1), you can conclude that A(2) = B(2). But now that leads to A(3) = B(3), and so on.

You prove the inductive step "whenever A(k) = B(k) then it follows that A(k+1) = B(k+1)," and then the theorem is proved generally.

In this case, we first show

sum nCi = 2^n

when n = 1, in other words, 1C0 + 1C1 = 2^1, which is certainly true, because 1C0 = 1, 1C1 = 1, and 2^1 = 2. [So the priming step is complete.]

[Now we have to show that IF WE ASSUME THAT sum kCi = 2^k, then
sum (k+1)Ci = 2^(k+1).]

So assume that sum kCi = 2^k.

Now sum (k+1)Ci = (k+1)C0 + (k+1)C1 + ... + (k+1)Ck + (k+1)C(k+1)

and you should be able to finish the algebra yourself and show that the right side boils down to 2^(n+1), as it is supposed to.

Now you argue that, since this is true regardless of the value of the integer k, by the Principle of Mathematical Induction, the formula sum
nCi = 2^n is true for all n.

This is quite an easy proof, but the idea of the principle of mathematical induction is a little difficult to understand for some people. I think I was about 17 when I first understood it, so you might be able to figure it out now. If you don't, in a couple of years you definitely will!

一般２項定理：ｎが実数αの場合

　ニュートンは、パスカルの三角形からこの一般二項定理を発見

$$
\begin{aligned}
(1+x)^\alpha &= 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\cdots \\
&= \sum_{k=0}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}x^k \qquad (|x|<1) \\
&= \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k \qquad (|x|<1)
\end{aligned}
$$

展開式を自在に使って，青年ニュートンは円周率を計算し，また(1 + x^2)^1/2を項別に積分して逆正弦関数arcsin x の展開式を求め，さらにそれを逆に解いて，西欧世界で最初に正弦関数sin x の展開式を求めた。

二項分布とは

ある集団において，特性 A を持つものの割合が p であり，持たないものの割合が q であるとする（ p + q = 1 ）。このとき，集団から無作為に n 人を抽出したとき，特性 A を持つものが x 人である確率

f (x) = nCx p^x q^(n - x),　　x = 0,1, … ,n,　p + q = 1

nCx=n!/[x!・(n-x)!]

2項展開と2項分布

二項展開から、

[p + (1-p)]^ n = nC0 p^0(1-p)^n + nC1 p^1(1-p)^(n-1)+ … + nCn p^n(1-p)^0　=1

これは、n回試行を行った場合の成功の回数が、0,1,2,・・・,nである確率を合計したもであり、当然、すべての成功する場合を尽くしているので、１に一致している。

フィボナッチ数列：漸化式

前の2項の和が次の項の値になるような数列です。

The nth Fibonacci number is the sum of the previous two Fibonacci numbers.

初期値　F(0)=0,F(1)=1

F(n) = F(n – 1) + F(n – 2)

- The first 21 Fibonacci numbers , also denoted as Fn, for n = 0, 1, 2, ... , 20 are:

0 1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 1597 2584 4181 6765

フィボナッチ数列：フィボナッチの兎の問題

In the West, the sequence was studied by Leonardo of Pisa, known as Fibonacci, in his Liber Abaci (1202). He considers the growth of an idealised (biologically unrealistic) rabbit population, assuming that:
In the "zeroth" month, there is one pair of rabbits (additional pairs of rabbits = 0). In the first month, the first pair begets another pair (additional pairs of rabbits = 1). In the second month, both pairs of rabbits have another pair, and the first pair dies (additional pairs of rabbits = 1). In the third month, the second pair and the new two pairs have a total of three new pairs, and the older second pair dies (additional pairs of rabbits = 2). The laws of this are that each pair of rabbits has 2 pairs in its lifetime, and dies.
Let the population at month n be F(n). At this time, only rabbits who were alive at month n − 2 are fertile and produce offspring, so F(n − 2) pairs are added to the current population of F(n − 1). Thus the total is F(n) = F(n − 1) + F(n − 2).

黄金数との関係

フィボナッチ数Fnのｎを大きくしていくと、続く2項の比　Fn+1/Fn　は、黄金数に限りなく近づく。

Limit　n--->∞　Fn+1/Fn　=　φ　=　(1+√5)/2　=1.61803

- Johannes Kepler observed that the ratio of consecutive Fibonacci numbers converges. He wrote that "as 5 is to 8 so is 8 to 13, practically, and as 8 is to 13, so is 13 to 21 almost”, and concluded that the limit approaches the golden ratio.

Fnは黄金数φで表わされる

Fn = [φ^n　-(1-φ)^n　]/√5**漸化式の例：フィボナッチ数列**

Example: Fibonacci numbers
Fn+2 = Fn+1 + fn with initial values F0=0,F1=0.

We obtain the sequence of Fibonacci numbers which begins:
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...
一般解の導出 The Fibonacci recursion

Fn+2 - Fn+1 - Fn =0
is similar to the defining equation of the golden ratio in the form
x^2 - X -1 = 0
which is also known as the generating polynomial of the recursion. By definition α,β is a root of the equation
α = (1+√5)/2,　β=(1-√5)/2.
Any root of the equation above satisfies x^2 - X -1 = 0 and multiplying by x^n　shows:
x^(n+2)-x^(n+1)-x^n = 0
Both α^n and β^n are geometric series (for n = 1, 2, 3, ...) that satisfy the Fibonacci recursion. Linear combinations of series α^n and β^n, with coefficients a and b, can be defined by
Fc(n)=a・α^n + b・β^n　for any real a and b.

All thus-defined series satisfy the Fibonacci recursion
Fc(n+2)=a・α^(n+2) + b・β^(n+2)

=a・(α^(n+1)+α^n) + b・(β^(n+1)+β^n)

=a・α^(n+1)+ b・β^(n+1) + a・α^n+b・β^n

=Fc(n+1) + Fc(n).
Requiring that Fc(0)=0,Fc(1)=1 yields
Fc(0)=a・α^0 + b・β^0 = a + b =0

Fc(1)=a・α^1 + b・β^1 = a・α + b・β =0

Then　a=1/√5　and　b=-1/√5. The solution of the Fibonacci recursion is
Fn=(α^n - β^n)/√5　=｛(1+√5)/2｝^n　＋　｛(1+√5)/2｝^n

α = (1+√5)/2,　β=(1-√5)/2.

•このことから　特性方程式の解の線形結合で、一般解が求められることが理解できる

ａ "(homogeneous) linear recurrence relation" is one like:

2*x_(n+3) + 5*x_(n+2) - x_(n+1) + 3*x_n = 0,

where you multiply each x_(n+i) value by some constant, any real

number, and you sum them all up and get zero. (Or you might have some
on the left side of the equation and some on the right, but that's the same thing.) Your second case is a linear recurrence relation, and they are rather simple: Change x_(n+i) into r^i and you will get a polynomial, like

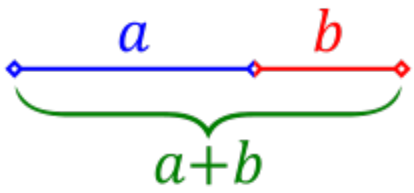
2*r^3 + 5*r^2 - r + 3 = 0,

find its roots (such as r1, r2, and r3), and x_n has the form

a1 * r1^n + a2 * r2^n + a3 * r3^n

where the constants a1, a2, a3 (up to the degree of the polynomial) are determined by the initial values (seeds, you called them) of the sequence. There is more math that can be said about this, but it gives you the idea.

Golden ratio
From Wikipedia, the free encyclopedia



$a+b$ is to a as a is to b

The golden section is a line segment divided according to the golden ratio: The total length a + b is to the longer segment a as a is to the shorter segment b.
In mathematics and the arts, two quantities are in the golden ratio if the ratio of the sum of the quantities to the larger quantity is equal to (=) the ratio of the larger quantity to the smaller one.

The golden ratio is an irrational mathematical constant, approximately 1.6180339887. The figure on the right illustrates the geometric relationship that defines this constant. Expressed algebraically:

$$\frac{a+b}{a} = \frac{a}{b} \equiv \varphi$$

This equation has as its unique positive solution the algebraic irrational number

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$$

Two quantities *a* and *b* are said to be in the *golden ratio φ* if:

$$\frac{a+b}{a} = \frac{a}{b} = \varphi.$$

One method for finding the value of φ is to start with the left fraction. Through simplifying the fraction and substituting in b/a = 1/φ,

$$\frac{a+b}{a} = 1 + \frac{b}{a} = 1 + \frac{1}{\varphi},$$

it is shown that,

$$1 + \frac{1}{\varphi} = \varphi.$$

Multiplying by *φ* gives

$$\phi + 1 = \phi^2$$

which can be rearranged to

$$\phi^2 - \phi - 1 = 0.$$

Using the quadratic formula gives the only positive solution as,

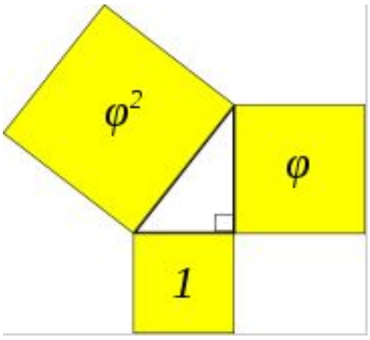
$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$$

Kepler triangle

A Kepler triangle is a right triangle with edge lengths in geometric progression. The ratio of the edges of a Kepler triangle are linked to the golden ratio

φ＝(1+√5)/２

and can be written: 1:√φ:φ , or approximately 1 : 1.2720196 : 1.6180339.



解説

Triangles with such ratios are named after the German mathematician and astronomer Johannes Kepler (1571–1630), who first demonstrated that this triangle is characterised by a ratio between short side and hypotenuse equal to the golden ratio. Kepler triangles combine two key mathematical concepts—the Pythagorean theorem and the golden ratio—that fascinated Kepler deeply, as he expressed in this quotation:

導出 Derivation

The fact that a triangle with edges 1, √φ and , φ forms a right triangle follows directly from rewriting the defining quadratic polynomial for the golden ratio :

φ^2＝φ＋１

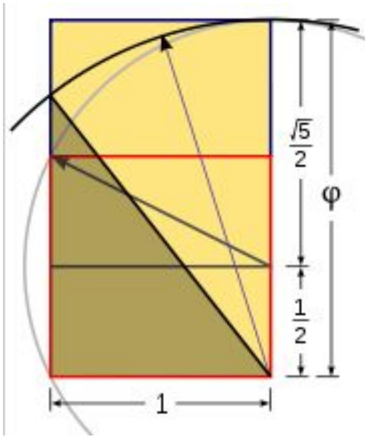
into Pythagorean form:

(φ)^2 = (√φ)^2 + (1)^2

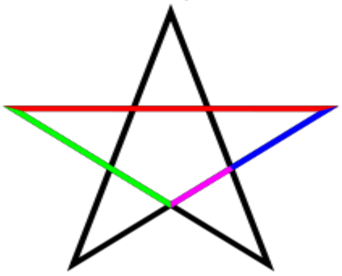
直角三角形(right triangle)において，hypotenuseは「斜辺」，adjacentは「底辺」，oppositeは「対辺」。アルキメデスの定理"The length of the hypotenuse of the right triangle is equal to the square-root of the sum of the squares of the other two sides (the adjacent sides). Assuming that the lengths of the other two sides are 8 [cm] and 5 [cm], the length of the hypotenuse is about 9.43398 [cm]."

作図方法

A Kepler triangle can be constructed with only straightedge and compass by first creating a golden rectangle:

- Construct a simple square
- Draw a line from the midpoint of one side of the square to an opposite corner
- Use that line as the radius to draw an arc that defines the height of the rectangle
- Complete the golden rectangle
- Use the longer side of the golden rectangle to draw an arc that intersects the opposite side of the rectangle and defines the hypotenuse of the Kepler triangle
- 

A pentagram colored to distinguish its line segments of different lengths. The four lengths are in golden ratio to one another.



黄金数とは

黄金分割や黄金四角形で現れる数φです。φ=(1+√5)/2

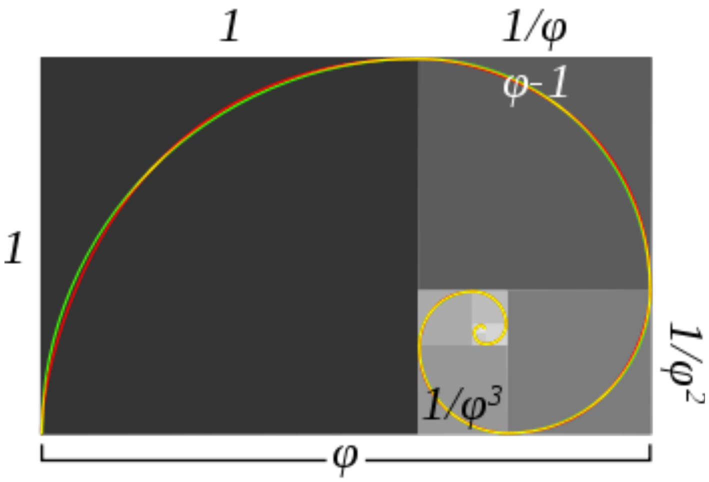
これは、ケプラーの三角形で紹介していますように、正方形の1辺の中点と斜め上の角点を結ぶ長さが√5/2ですので、簡単にコンパスで黄金四角形を作成できます。

そして、この黄金四角形の対角線と底辺の角度は　黄金角度と呼ばれています。

黄金比の螺旋

原点からの長さが幾何級数的に増大する螺旋を対数螺旋とか等角螺旋という。この特殊な場合が黄金螺旋である。

ｆ(ｎ)＝φ^n／√５　φ＝（１＋√５)／２は指数関数で、対数らせんの式に変換できる。これが、黄金分割らせんのできる理由である。



A Fibonacci spiral which approximates the golden spiral, using Fibonacci sequence square sizes up to 34.

植物は黄金角で葉（種）を作っているようです。葉の回転角が円周をピッタリ黄金比に分けるような植物があったとすると, その回転角は137:507度の黄金角度になる

Polynomial

多項式（たこうしき、*polynomial*）は**定数**および**変数**の和と積のみからなり、**代数学**の重要な対象となる**数学**的概念である。歴史的にも現代代数学の成立に大きな役割を果たした。 多項式とは

$3x^3 - 7x^2 + 2x - 23$

のような形をした式である。加法や減法を全て加法として次の式のように考えた場合、

$3x^3 + (-7x^2) + 2x + (-23)$

加法の記号で区切られた式の "$3x^3$", "$-7x^2$", "$2x$", "-23" のことを項（ こう、*term*）と呼び、複数の項を足し合わせることでできる式であることから多項式と呼ばれる

From Wikipedia, the free encyclopedia

In mathematics, a polynomial is an expression of finite length constructed from variables (also known as indeterminates) and constants, using only the operations of addition, subtraction, multiplication, and non-negative integer exponents. For example, $x^2 - 4x + 7$ is a polynomial, but $x^2 - 4/x + 7x^{3/2}$ is not, because its second term involves division by the variable x (4/x) and because its third term contains an exponent that is not an integer (3/2).

Polynomial functions

A **polynomial function** is a function that can be defined by evaluating a polynomial. A function f of one argument is called a polynomial function if it satisfies

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

for all arguments x, where n is a non-negative integer and a_0, a_1,a_2, ..., a_n are constant coefficients.

For example, the function f, taking real numbers to real numbers, defined by

$$f(x) = x^3 - x$$

is a polynomial function of one argument.

Polynomial equations

A polynomial equation, also called **algebraic equation**, is an equation in which a polynomial is set equal to another polynomial.

$$3x^2 + 4x - 5 = 0$$

is a polynomial equation.

虚数　i　の歴史

虚数i：ニュートン（1642〜1727 年）、デカルト、オイラー時代には、–1 はやはり代数的虚構とされていたのである」（Cajori 1970, p.357）。

カルダノは２次方程式の解法「*x* + *y* =*10*、*x*- *y* =*40* を満たす*x*、*y* を求めよ」で、言葉を残しています。「それによって受ける**精神**的**苦**

痛は忘れ，ただこれらの量（根号内が負の解）を導入せよ」．こうして，2乗して負になる新しい「数」が数学上の概念として導入されていきました。

そして次のようにつづけたのである。**「精神的な苦痛を無視すれば、**この二つの数のかけ算の答は４０となり、確かに条件を満たす。」こうしてカルダノは、虚数をもちだせば、答えのない問題にも答えが出せることをはじめて示した。だがカルダノは、次のように書き添えてもいる。**「これは詭弁的であり、数学をここまで精密化しても実用上の使い道はない」。**

マイナスの数の平方根は、虚数(nombre imaginaire)とデカルトに呼ばれ、なかなか市民権を得ませんでしたが、デンマークの測量技師カスパー・ヴェッセルにより図で可視化された

虚数の　i記号は、レオナルド・オイラーにより1777年に導入された。数学への複素平面の導入は、カスパー・ヴェッセル（1745-1818）によりなされ劇的に数の概念を拡張した

Complex Number and Euler's formula

In complex analysis the complex numbers are customarily represented by the symbol *z*, which can be separated into its real (*x*) and imaginary (*y*) parts, like this:

$$z = x + iy$$

where *x* and *y* are real numbers, and *i* is the imaginary unit. In this customary notation the complex number *z* corresponds to the point (*x*, *y*) in the Cartesian plane.

In the Cartesian plane the point (*x*, *y*) can also be represented in polar coordinates as

$$(x, y) = (r\cos\theta, r\sin\theta) \qquad \left(r = \sqrt{x^2 + y^2}; \quad \theta = \arctan\frac{y}{x}\right).$$

In the Cartesian plane it may be assumed that the arctangent takes values from $-\pi$ to π (in radians), and some care must be taken to define the *real* arctangent function for points (*x*, *y*) when $x \leq 0$. In the complex plane these polar coordinates take the form

$$z = x + iy = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$$

where

$$|z| = \sqrt{x^2 + y^2}; \quad \theta = \arg(z) = -i \log \frac{z}{|z|}.$$

Here $|z|$ is the *absolute value* or *modulus* of the complex number z; θ, the *argument* of z, is usually taken on the interval $0 \leq \theta < 2\pi$; and the last equality (to $|z|e^{i\theta}$) is taken from Euler's formula. Notice that the *argument* of z is multi-valued, because the complex exponential function is periodic, with period $2\pi i$. Thus, if θ is one value of $\arg(z)$, the other values are given by $\arg(z) = \theta + 2n\pi$, where n is any integer $\neq 0$

Euler's identity

Jacob Bernoulli contributed to the history of e in 1683 when he was looking for a way to compound interest. He tried finding the limit of (1+(1/n)), where n is usually infinity. This was the first approximation of e. Shortly after this the number e finally appeared. A letter written to Huygens had the notation b, which is now known as e (O'Connor). 最初のe は、金利計算から求められた。

A famous mathematician named Leonhard Euler was one of the main contributors in history to the field of mathematics. Euler first used the symbol for pi as the ration of the circumference to the diameter in a circle and was also the first to use i to be equal to square root of -1. He was also the first to **find** that e^(pi*i)+1=0. Euler was born in Basel and was taught by another famous mathematician, Bernoulli.

Euler was especially important to the function e. Euler was the first person to use the notation e, which he used in a letter he wrote to a man named Goldbach. A book, "Introductio in Analysin Infinitorum", was written by Euler in **1748** that describes his views and findings about e. In **Euler's** work was the **formula** for e, which is

"$e = 1 + {}^1/_{1!} + {}^1/_{2!} + {}^1/_{3!} + \ldots$". Euler was able to use his own **formula** to **find** e to 18 decimal places. His approximation for e was 2.718281828459045 (Singer).

The exponential function e^z can be defined as the limit of $(1 + z/N)^N$, as N approaches infinity, and thus $e^{i\pi}$ is the limit of $(1 + i\pi/N)^N$. In this animation N takes various increasing values from 1 to 100. The computation of $(1 + i\pi/N)^N$ is displayed as the combined effect of N

repeated multiplications in the complex plane, with the final point being the actual value of $(1 + i\pi/N)^N$. It can be seen that as N gets larger $(1 + i\pi/N)^N$ approaches a limit of −1.

In mathematical analysis, **Euler's identity**, named after Leonhard Euler, is the equation

$$e^{i\pi} + 1 = 0,$$

where

e is Euler's number, the base of the natural logarithm,
i is the imaginary unit, one of the two complex numbers whose square is negative one (the other is $-i$), and
π is pi, the ratio of the circumference of a circle to its diameter.

Euler's identity is considered by many to be remarkable for its mathematical beauty. Three basic arithmetic operations occur exactly once each: addition, multiplication, and exponentiation. The identity also links five fundamental mathematical constants:

- The number 0.
- The number 1.
- The number π, which is ubiquitous in trigonometry, geometry of Euclidean space, and mathematical analysis ($\pi \approx 3.14159$).
- The number *e*, the base of natural logarithms, which also occurs widely in mathematical analysis ($e \approx 2.71828$).
- The number *i*, imaginary unit of the complex numbers, which contain the roots of all nonconstant polynomials and lead to deeper insight into many operators, such as integration.

Furthermore, in mathematical analysis, equations are commonly written with zero on one side.

Euler's identity is a special case of the more general identity that the nth roots of unity, for $n > 1$, add up to 0:

$$\sum_{k=0}^{n-1} e^{2\pi ik/n} = 0.$$

Euler's identity is the case where $n = 2$.

一番易しいと思われる証明

（証明）
３つの数　$e^{i\theta}$、$\cos\theta$、$i\cdot\sin\theta$　が一次従属であることを示す。
$A e^{i\theta} + B\cos\theta + C\cdot i\cdot\sin\theta = 0$　（A、B、C は定数）
において、θに関する微分を繰り返して、

$$i\cdot A e^{i\theta} - B\sin\theta + C\cdot i\cdot\cos\theta = 0$$

$$-A e^{i\theta} - B\cos\theta - C\cdot i\cdot\sin\theta = 0$$

このとき、A、B、C に関する３元連立方程式の係数行列の行列式を求めると、0になる。よって、３つの数　$e^{i\theta}$、$\cos\theta$、$i\cdot\sin\theta$　は一次従属である。
$A\neq 0$　として、　$e^{i\theta} = a\cos\theta + b\cdot i\cdot\sin\theta$　（a、b は定数）　と表される。
　$\theta=0$　とおくと、　$a=1$
上式を微分して、

$$i\cdot e^{i\theta} = -a\sin\theta + b\cdot i\cdot\cos\theta$$

$\theta=0$　とおくと、　$b=1$
したがって、　$e^{i\theta} = \cos\theta + i\cdot\sin\theta$　　（証終）

円分多項式　cyclotomic polynomial

円周等分多項式或いは円分多項式と呼ばれる次の方程式、

$x^n - 1 = 0$
は、任意の n について、ベキ根と四則演算だけで解を求めることができる。

n=2 の場合
根の公式
n=3 の場合
根の公式

n=4　の場合
明らか
n=5 の場合
１の５乗根。黄金数が出てくる。
n=7 の場合
３次方程式 カルダノの公式
n=17 の場合
２次方程式だけで解ける
計算方法は大変

$x^5=1$を解く

因数分解による方法

因数分解して

$(x-1)(x^4+x^3+x^2+x+1)=0$

左辺より$x=1$
右のカッコ内を$x^2\neq0$に注意してx^2で割ると$x^2+x+1+1/x+1/x^2=0$となる。
これは相反方程式なので整理すると$\{(x+1/x)^2\}-2+(x+1/x)+1=0$
$x^2+x=t$として整理すると$t^2+t-1=0$
因数分解して$t=(-1\pm\sqrt{5})/2$となる。
$x+1/x=t$に戻して

$x^2-((-1\pm\sqrt{5})/2)x+1=0$
$2x^2-(-1\pm\sqrt{5})x+2=0$
再び解の公式に入れて $x=[(-1+\sqrt{5})\pm\sqrt{(-10-2\sqrt{5})}]/4, [(-1-\sqrt{5})\pm\sqrt{(-10+2\sqrt{5})}]/4.$
で, $2\sqrt{5} < 10$ だから, 虚数単位 i を用いると
$x=[(-1\pm\sqrt{5})\pm i\sqrt{(10\pm2\sqrt{5})}]/4$
($\sqrt{5}$ の直前の複号だけ同順) となる。

ドモアブルの公式を使う方法

$x=\cos\theta+i\cdot\sin\theta$ とおく。

$x^5=\cos5\theta+i\cdot\sin5\theta$となる。

$\cos5\theta=1$より、$5\theta=0+2n\pi$(nは整数)となる。$\theta=2n\pi/1$　n=1,2,3,4.5　の場合が解。

ド・モアブルの公式

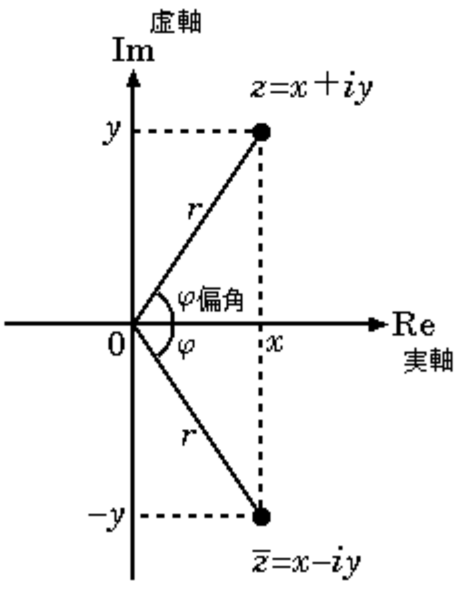
整数nに対して、

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\cdot\sin\theta$

が成り立つという複素数に関する定理である

アブラーム・ド・モアブル（Abraham de Moivre, 1667 - 1754年）はフランスの数学者である。

ガウス平面



一つの複素数 $x + iy$ は二つの実数 x, y の組 (x, y) によって特徴付けられる。一方で二つの実数の組はデカルト座標を敷いた平面上の点として特徴付けられる。そこで、複素数を平面上の点と一対一に対応付けることによって、複素数をその内部の点として含む平面を考えることができる。このようにして得られる平面を、**ガウス平面**

(*Gaussian plane*) あるいは**複素平面**（ふくそへいめん、*complex plane*）などとよぶ。ガウス平面では、*x* 座標に実部、*y* 座標に虚部が対応し、 *x* 軸のことを**実軸** (*real axis*)、 *y* 軸のことを**虚軸** (*imaginary axis*)と呼ぶ。

複素数 *z*, *w* に対して

$$d(z,\ w) = |z - w|$$

によって距離を定めれば **C** は距離空間となる。この距離は、ガウス平面上で考えると、複素数が普通のユークリッド平面上の点と同じように扱えることが分かる。ガウス平面は複素数の形式的な計算を視覚的に見ることができ、数の概念そのものを拡張した。

極形式

ガウス平面を利用すると、複素数の極座標による表示として**極形式** (*polar form*) で表示できる。複素数$z = x + iy$に、ガウス平面上の点$(x,\ y)$を対応させたとき、この点が極座標で$(r,\ \theta)$と表されるなら

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2 + y^2},$$
$$\theta = \begin{cases} \arctan(y/x) & (\text{if } x > 0) \\ \pi/2 & (\text{if } x = 0, y > 0) \\ 3\pi/2 & (\text{if } x = 0, y < 0) \\ \arctan(y/x) + \pi & (\text{if } x < 0) \end{cases}$$

が成り立つ。*r*は*z*の**絶対値** ($r = |z|$) である。θを**偏角** (*argument*) といい、「arg *z*」と書く。$z = 0$の時の偏角は任意の実数とする。なお、偏角の定義式はarctan(*y* / *x*)だけが書かれていることが多いが、実際には実部が0でないとは限らないし、偏角の範囲は半回転に留まらないため上記のような場合分けが必要である。

偏角θの単位をラジアンとするならば、これらの関係式とオイラーの公式から

$$\begin{aligned} z &= x + iy \\ &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

という表示が得られる。

$r(\cos\theta + i\sin\theta)$あるいは$re^{i\theta}$のような複素数の表示を**極形式**といい、$re^{i\theta}$のような表示は**オイラー表示**とも呼ぶ

Dividing by Complex Numbers

Date: 05/04/2003 at 23:07:51
From: Ryan
Subject: Rational expressions with imaginary numbers

Hi,

I'm difficulty with problems that have an imaginary number but don't cancel. Example: Divide each pair of complex numbers: (8+4i)/(1+2i)

Any help would be great.
Thanks.

Date: 05/05/2003 at 05:02:42
From: Doctor Luis
Subject: Re: Rational expressions with imaginary numbers

Hi Ryan,

Good question. There's a trick for dividing by complex numbers, and to use it you need to understand something called the conjugate complex number.

Essentially, the conjugate of a complex number is the number you get when you replace (i) by (-i). For example, the conjugate of 1+3i is 1+3(-i)=1-3i, and also, the conjugate of -3-2i is -3-2(-i)=-3+2i

Now, something funny happens when you multiply a complex number by its conjugate. The answer turns out to be a real number. I'll illustrate with 1+3i and its conjugate 1-3i

(1+3i)*(1-3i) = 1^2 - (3i)^2 = 1 - (9 * (-1)) = 1 + 9 = 10

Here, I used the algebraic formula (a-b)(a+b) = a^2 - b^2 to multiply them. (Don't forget that i^2 = -1)

You should verify for yourself that for any complex number z=x+iy and its complex conjugate z'=x-iy, the product z*z' is a real number (and that it equals x^2+y^2).

Knowing this fact about complex numbers, to divide you simply multiply and divide by the conjugate of the denominator.

Here's how:

```
20 + 30i          20 + 30i          -1 - 2i
--------- = ---------- * ---------- (conjugate trick)
-1 + 2i           -1 + 2i           -1 - 2i
```

```
(20 + 30i)(-1 - 2i)
= ------------------- (multiplying bottom)
(-1)^2 + (2)^2
```

= (40 - 70i)/5 (after multiplying top)

= 8 - 14i (final answer)

That's all there is to it. You make the denominator into a number you can divide by (that is, a real number), using complex conjugates. With this background, you should be able to solve the division you asked about.

Let us know if you have any more questions.

DeMoivre's Formula

From: Greg Lukens

Subject: Demove's Formula

I have forgotten about DeMoivre's formula. Thanks in advance for the reminder.

Greg Lukens

From: Doctor Benway
Subject: Re: Demove's formula

Hi Greg,

I believe the formula you're looking for is DeMoivre's formula, which is the following:

(cos(theta) + i*sin(theta))^n = cos(n*theta) + i*sin(n*theta)

This formula is useful when you have a complex number and want to raise it to some power without doing a lot of work.

If all you want is the formula, you can ignore the rest of this message. However, if you want a little more insight into what is going on, read on.

Recall that any complex number can be written in the form r*e^(i*theta). If you plot a complex number in the complex plane (where the x-axis is the real axis and y-axis is the imaginary axis), then "r" will be the distance from the point to the origin and theta will be the angle a line from the origin to the point makes with the x-axis. A little trig shows that a complex number written as r*e^(i*theta) can also be written as r*cos(theta)+r*i*sin(theta).

Knowing this little fact gives us the ability to switch back and forth between ways of writing complex numbers, depending on what we want to do with them. If we want to add complex numbers, then the form a + b*i is easiest, whereas if we want to multiply them together, it is easier to use the form r*e^(i*theta).

Essentially what you are doing is taking a complex number of the form (a + b*i), converting it to the form r*e^(i*theta), raising it to a power in that form, then converting back to the first form. Observe:

(r*cos(theta) + r*i*sin(theta))^n
= (r*(cos(theta) + i*sin(theta))^n
= (r^n) * (cos(theta) + i*sin(theta))^n
= (r^n) * (e^(i*theta))^n
= (r^n) * (e^(n*i*theta))
= (r^n) * (cos(n*theta) + i*sin(n*theta))

Of course knowing DeMoivre's formula allows us to go straight from

(r*(cos(theta) + i*sin(theta))^n

to

(r^n) * (cos(n*theta) + i*sin(n*theta)).

Thanks for writing, hope this helps.

第6回 極限とは。2項定理から、ネイピア数ｅ　を求める。連続複利計算とe^x

指数関数y=e^xの微分は何故e^xか。　オイラーの公式と等式（eˆiπ＋1＝0）。

Derivative of e^x

From: Reb
Subject: Proof of the derivative of e^x
Hi Doctor Math,
I was wondering if you could tell me how to prove that the derivative of e^x is e^x. I need a step by step proof.
Thanks a lot.
Subject: Re: Proof of the derivative of e^x
In the 1730's Euler investigated the result of compounding interest continuously when a sum of money, say, is invested at compound interest.
If interest is added once a year we have the usual formula for the amount, A, with principal P, rate of interest r% per annum, and t the time in years:
A = P(1 + r/100)^t
If interest were added twice a year, we would replace r by r/2 and t by 2t. The formula would become:
A = P(1 + r/(2x100))^(2t)
If interest were added three times a year, then at the end of t years A would be:
A = P(1 + r/(3x100))^(3t)

and if we added interest N times a year, then after t years the amount
A would be
$A = P[1 + r/(N\times100)]^{(Nt)}$
Now to simplify the working we let $r/(100N) = 1/n$, so $N = nr/100$ and
$A = P[1 + 1/n]^{(nrt/100)}$
$= P[(1 + 1/n)^n]^{(rt/100)}$
We now let $n \to$ infinity and we must see what happens to the expression $(1 + 1/n)^n$ as n tends to infinity.
Expanding by the binomial theorem
$(1 + 1/n)^n = 1 + n(1/n) + n(n-1)/2!\,(1/n)^2 +$
$n(n-1)(n-2)/3!\,(1/n)^3 + ...$
Now take the n's in $1/n^2, 1/n^3, ...$ in the denominators and distribute one n to each of the terms n, n-1, n-2, ... in the numerator, getting
$1 \times (1-1/n) * (1-2/n) * ...$
so we now have
$(1 + 1/n)^n = 1 + 1 + 1(1-1/n)/2! + 1(1-1/n)(1-2/n)/3! + ...$
Now let $n \to$ infinity and the terms $1/n, 2/n, ...$ all go to zero, giving
$(1 + 1/n)^n = 1 + 1 + 1/2! + 1/3! + 1/4! + ...$
and this series converges to the value we now know as e.
If you consider e^x you get
$(1 + 1/n)^{(nx)}$
Expand this by the binomial theorem and you have
$(1 + 1/n)^{(nx)} = 1 + (nx)(1/n) + nx(nx-1)/2!\,(1/n)^2 + ...$
and carrying through the same process of putting the n's in the denominator into each term in the numerator, as described above, you
obtain
$e^x = 1 + x + x^2/2! + x^3/3! + ...$
and differentiating this we get
$d(e^x)/dx = 0 + 1 + 2x/2! + 3x^2/3! + ...$
$= 1 + x + x^2/2! + x^3/3! + ...$
$= e^x$
Reverting to our original problem of compounding interest continuously, the formula for the amount becomes
$A = P.e^{(rt/100)}$

You might like to compare the difference between this and compounding
annually.
If P = 5000, r = 8, t = 12 years
Annual compounding gives
A = 5000(1.08)^12 = 12590.85
Continuous compounding gives
A = 5000.e^(96/100) = 13058.48
The difference is not as great as might be expected

Proof of e^(ix) = cos(x) + isin(x)
From: Walter Graf
Subject: Proof of e^(ix) = cos(x) + isin(x)

In the equation e^(iPi) - 1 = 0, the proof is to evaluate
e^(ix) = cos(x) + isin(x) for x = Pi.
I would like to see a rigorous proof of of the the above equation.

Thank you,
From: Doctor Mitteldorf
Subject: Re: Proof of e^(ix) = cos(x) + isin(x)

Dear Walter,

This is called the **Euler equation**, and it's not something you can prove rigorously. It's a definition, and I'd like to convince you that it's the only sensible definition, of how to compute imaginary exponentials.

I can think of three approaches to verifying the Euler equation, but unfortunately one of them is all calculus, one uses calculus explicitly, and only the third is free of calculus. I'm just guessing from your age that you may not have studied calculus yet.

You can verify that the Euler equation makes a sensible definition by expanding the two sides as Taylor series in x. You can also differentiate both sides and see that the answer is self-consistent. Thirdly, you can use the formula for cos(2x) and sin(2x) to show that the right side has the property you expect from an exponential, so that e^i(2x) = (e^ix)^2.

So start with choice 3. You have the formulas

$\cos(2x) = \cos^2(x) - \sin^2(x)$ and
$\sin(2x) = 2 \sin(x) \cos(x)$

You'd also want to demand that $e^{i(2x)} = (e^{ix})^2$. That means that your new definition of e^{ix} is behaving like an exponential. See if you can put these together to show that

$e^{i(2x)} = \cos(2x) + i \sin(2x)$.

The Taylor expansion is something you can appreciate without calculus, although its roots are in calculus. It's a series expression for a function. You may have run across the following infinite series representations of cos and sin and e^x. In fact, this is the most straightforward way to compute the value of $\sin(x)$ or e^x for any given x.

$\cos(x) = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots$
$\sin(x) = x - x^3/3! + x^5/5! - x^7/7! + \ldots$
$e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + \ldots$

The ! in these equations means factorial. In other words,
$4! = 4*3*2*1$.

See if you can use these infinite series expressions to verify the Euler equation.

After you complete these two projects, I'm hoping you'll find the Euler equation very plausible. If you still want more proof, write back again...

Euler Equation and DeMoivre's Theorem

From: Anthony
Subject: e, pi, and i

Is there a proof of e^(i*Pi) +1 = 0?

From: Doctor Anthony
Subject: Re: e, pi, and i

The derivation is not too difficult if you are familiar with the basics of complex numbers and exponential functions.

Start with z = cos(x) + i*sin(x) ......(1)

Then dz/dx = -sin(x) + i*cos(x)
= i(cos(x) + i*sin(x)) (since i^2 = -1)
= i*z
So dz/z = i*dx

Now integrate both sides

ln(z) = i*x + c From (1); when x=0, z=1 so c=0
ln(z) = i*x
z = e^(i*x) but z = cos(x) + i*sin(x), So
cos(x) + i*sin(x) = e^(i*x) ......(2)
Put x = pi in (2) and we get:
-1 + 0 = e^(i*pi)
and so e^(i*pi) + 1 = 0
This is the Euler equation.

Now returning to (2) we have
[cos(x)+i*sin(x)]^n = [e^(i*x)]^n
= e^(i*nx)
= cos(nx) + i*sin(nx)
and this is the statement of DeMoivre's theorem.
第7回 極限と微分。解析解を求める。

表面積一定で容積最大となるアイスクリームのコーンの形状は？

容積一定で表面積を最大とする缶の形状は？。

Ice Cream Cone Problem

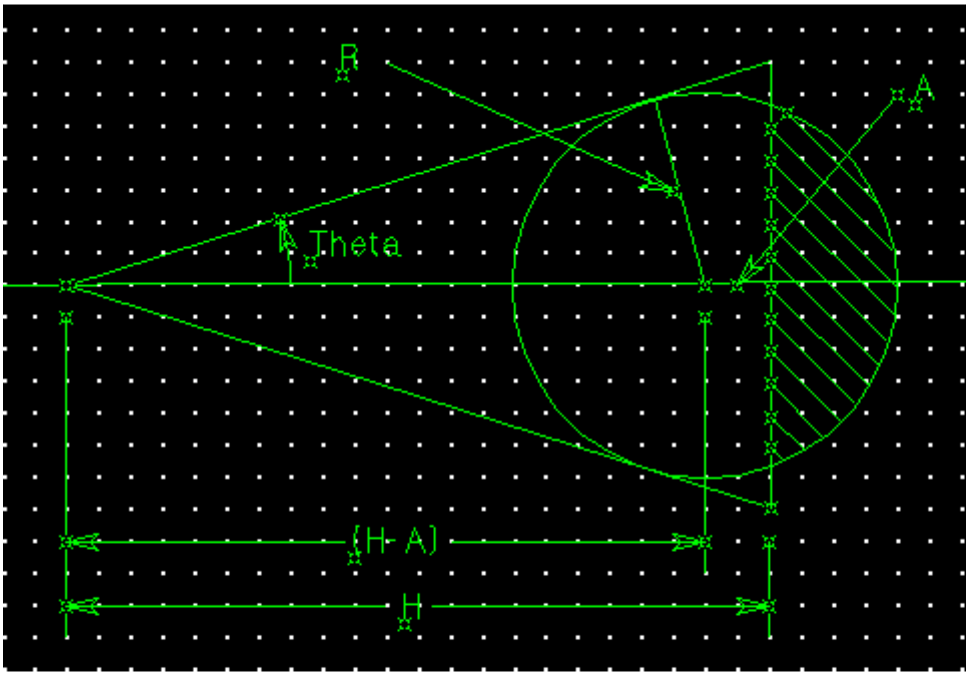
Date: 02/27/2000 at 19:57:39From: Tripp RatcliffSubject: Maximum volumeI'm not sure if you've heard of the ice cream cone problem, but it goes like this: You are to place a sphere of ice cream into a cone of height 1. What radius of the sphere will give the most volume of ice cream inside the cone (as opposed to above the cone) for a cone with a base angle of 30 degrees? I can not figure out how to solve it. Any suggestions?

Dear Tripp,

Here is one possible solution to this problem. Some of the details have been deliberately left out so you will have to go through this pretty carefully to be sure I haven't made any mistakes and to also be sure you understand this solution. You may write back if you still have any questions.

RESTATEMENT OF THE CONE PROBLEM: Find the size of a perfect sphere of
ice cream that will result in the most volume of ice cream within a perfect cone. For this particular case, let the cone height be H and the cone base angle be 30 degrees.

First, one may assume that the sphere that satisfies these conditions must lie only partially inside and partially outside of the cone. In the extreme, a very tiny sphere near the bottom of the cone will certainly occupy less volume then a slightly larger sphere. So the sphere volume within the cone will increase with the sphere's radius until, at some point, the radius becomes so large that most of the sphere volume will lie outside of the cone. However, as long as the center of the sphere lies at, below, or only slightly above an imaginary plane covering the top of the cone, there is a simple relation between the sphere's radius (R), the cone height (H), and the perpendicular distance between the sphere's center and the plane covering the top of the cone (A). In referring to the following figure:



it may be seen that this relation is:

$R = (H - A)[\sin(\text{Theta})]$,

where: A is positive if measured between the imaginary plane and the inside of the cone, and negative if measured from the imaginary plane and outside of the cone, and Theta equals (30/2 = 15 degrees). This expression for R is only valid for values of A between +H and ca. -0.0718H. Do you see why?

Since, under these conditions, it is possible to find an expression for the volume of the sphere that lies inside the cone (as a function of the sphere's radius and the perpendicular distance between the sphere's center and the plane surface covering the top of the cone), it is also possible to set the first derivative of this expression equal to zero to see if a maximum can be found. Note: H and Theta are both constants.

One next needs to find the sphere volume within the cone, as a function of R and A. One way to obtain this expression is to find the sphere volume outside of the cone, then subtract that result from the total sphere volume. If one allows the sphere volume lying outside of the cone to be represented by Vout, it can be shown that:

Vout = Pi[(2/3)R^3 - AR^2 + A^3/3].

This expression is a result of integrating the expression:

dVout = Pi[y^2]dx = Pi[R^2 - x^2]dx,

between the x limits of A and R.

Subtracting Vout from the total sphere volume of (4/3)Pi R^3, to produce an expression for the sphere volume within the cone (Vin), produces the following:

Vin = Pi[(2/3)R^3 + AR^2 - A^3/3].

Taking the derivative of Vin, with respect to A, produces:

(dVin/dA) = Pi[2R^2(dR/dA) + R^2 + 2AR(dR/dA) - A^2].

Rearranging slightly produces:

(dVin/dA) = Pi[2R(R + A)(dR/dA) + (R + A)(R - A)]
= Pi(R + A)[2R(dR/dA) + (R - A)].

Setting the last expression above for (dVin/dA) equal to zero and solving for A produces:

Pi(R + A)[2R(dR/dA) + (R - A)] = 0.

But, Pi(R + A) cannot equal zero (as long as A is not is too negative), so:

[2R(dR/dA) + (R - A)] = 0.

If solving the expression above for A produces a reasonable value for A (i.e., a value for A between +H and ca. -0.718H, then a maximum sphere volume within the cone (for that value of A) will have been found.

Note that (dR/dA) = [-Sin(Theta)], from the first equation written above. Using that first expression for R and (dR/dA) = [-Sin(Theta)] in the equation above should produce the following result for A:

A = [H Sin(Theta)][1.0 - 2 Sin(Theta)]/[1.0 + Sin(Theta) -
2 Sin^2(Theta)].

For H = 1.0 and Theta = 15 degrees, A is approximately = +0.111, or about 11% of H.

Minimizing the Surface Area of a Can

From: Chris Donges
Subject: Surface area
What coke can dimensions would use the least amount of aluminum possible while still holding 375 ml? I'm not sure where to start - please help.
From: Doctor Jerry
Subject: Re: Surface area
Hi Chris,
I'll have to make some assumptions. If the coke can can be thought of as a cylinder with two circular ends, then you can write two formulas:
V = pi*r^2*h
and
S = 2*pi*r^2 + 2*pi*r*h.
The second is the areas of the two ends plus the area of the cylinder.
If r and h are in cm, you can convert 375 ml into cubic centimeters and then you'll have an equation like:
375*k = 2*pi*r^2 + 2*pi*r*h.
You can solve this for h in terms of r. Substitute this into the formula V = pi*r^2*h, which now will be entirely in terms of r. Now, just maximize V. You can do this with calculus or graphing.
Cylinder Problem
From: John Van Straalen
Subject: Geometry cylinder problem
The following question was brought up in my math class concerning the volume and surface are of a can.
Given an aluminum soft drink can with radius 3.25 cm, height 12 cm, volume 398.2 cubic cm, and surface area 311.4 square cm, is it possible to construct a can with a larger volume but with the same surface area? Can you construct a can with a smaller surface area but the same volume?
Is there a way to find the dimensions of the can with the largest volume but with the same surface area? Can you find the dimensions of the can with the smallest surface area but the same volume?

Any help, hints, or formulas that would help me answer these questions would be appreciated.
From: Doctor Jerry
Hi John,
These questions often come up in calculus. They can be solved by graphing, although you may have to be content with an approximate answer.
Suppose the volume of the can is fixed and we want to choose the dimensions so that the surface area is a minimum.
So, V = pi*r^2*h, where V is fixed. Note that this forces r and h to vary so that the product r^2*h is always equal to V/pi.
Surface area S = 2pi*r*h+2pi*r^2 = lateral surface plus the two ends.
Now, from the fact that r^2*h = V/pi, we can solve for h = V/(pi*r^2).
So,
S = 2pi*r*V/(pi*r^2)+2pi*r^2
Now you can graph S as a function of r and choose the low point.
Looking at the graph, is there a high point, that is, is there a maximum surface area for a given volume?

第8回　方程式の多項式近似（マクローリン展開、テーラー展開）。

sin(Θ)を求める。

マクローリン展開とは　Maclaurin expansion

関数f(x)をマクローリン級数で表すことを、マクローリン展開 (Maclaurin expansion) という。

マクローリン展開は、原点x=0の周りでのテーラー展開である。

マクローリン展開は、関数を　べき級数(x^n　の荷重和)の形　で近似したものである。

$$f(x) \simeq f(0)+f'(0)\,x+\frac{f''(0)}{2}\,x^2+\frac{f^{(3)}(0)}{3!}\,x^3+\frac{f^{(4)}(0)}{4!}\,x^4+\cdots+\frac{f^{(n)}(0)}{n!}\,x^n+\cdots$$

係数は何故　1/n！*fのn回微分　なのか

いまxが零の周りの十分小さい範囲で変化するものとして、f(x)の近似値をもとめるとする。

すぐ気付くのは、xが十分に小さいのですから

$$f(x) \simeq f(0)$$

のようにf(x) を定数にしてしまうという近似だ。実際、場合によっては、これも十分に「使える」近似だと思われる。しかし、x が変化するときf(x) も変化するはずだから、そのことも取り入れた近似をした方がいいだろう。そこで、微分の定義式で でx0 を0 とおき、微少量dx をx とおく ことで、

$$f(x) \simeq f(0) + f'(0)\,x$$

のように一次式で近似することです。 、さらに、二次式、三次 式と多項式の次数を上げながら、よりよい近似を作っていくことができるはずだ。そこで、ためしに

$$f(x) \simeq a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + a_5\,x^5 + \cdots$$

のような（べき級数の形の）近似があると仮定する。 係数a0, a1, a2, . . . をどうやって決めればいいかを考えよう。

まず、x=0を代入してみれば a0=f(0)が得られる。

つぎに、1階微分を行う。

$$f'(x) \simeq a_1 + 2a_2\,x + 3a_3\,x^2 + 4a_4\,x^3 + 5a_5\,x^4 + \cdots$$

ここで x=0 を代入すれば再び右辺はほとんど消えて、f'(0) = a1 が得られる。こうして、a0, a1 は前に書いた 一次近似と一致するように決まった。

さらにもう1階微分すれば、x=0を代入して f''(0)=2・a2 から a2=f''(0)/2 が得られる。これで2次式でx=0に近いxが近似できた。

同様に、何回も微分して x=0 とおいて、係数を求めることができる。

この考えで、うまくいくことを、一般形で示そう。まずxのm乗のn回微分は、下記のようになる。

$$\frac{d^n}{dx^n}x^m = \begin{cases} \frac{m!}{(m-n)!}x^{m-n} & n \le m \\ 0 & n > m \end{cases}$$

これは、繰り返し微分することで理解できるでしょう。特に、xのn乗のn回微分は、n！です。ここで、前の係数anの付いた近似式をn回微分してみると　n − 1 番目までの項が消えて、

$$f^{(n)}(x) \simeq n!\,a_n + (n+1)!\,a_{n+1}\,x + \frac{(n+2)!}{2!}\,a_{n+2}\,x^2 + \cdots$$

のようになります。ここでx = 0 とすれば、今度はn+1 番目よりあとの項が消えて、

f^n(0)=n!・an

となるので、係数anは字式で決定できる。

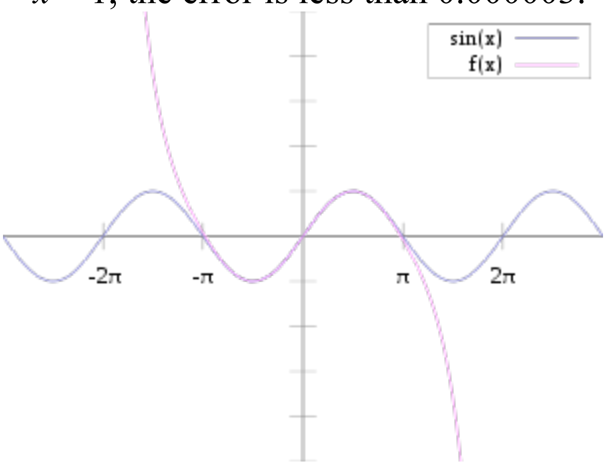
$$a_n = \frac{f^{(n)}(0)}{n!}$$

これを　前の近似式に代入することで、マクローリン展開が得られる。

Pictured on the right is an accurate approximation of sin(x) around the point $x = 0$. The pink curve is a polynomial of degree seven:

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

The error in this approximation is no more than $|x|^9/9!$. In particular, for $-1 < x < 1$, the error is less than 0.000003.



テーラー展開の場合

テーラー展開は、x=0の周りの近似のマクローリン展開と異なる。しかしx=x0の周りでの近似であるので、この場合と同様である。 微分する場所がx=x0であること、f(x)がf(x-x0)であることであるので、xの代わりにx-xoをfの原点でのn回微分 f^n(0)の代わりに f^n(x0) マクローリン展開に代入すれば、テーラー展開式が求められる。

Several important Maclaurin series expansions follow. All these expansions are valid for complex arguments *x*.

Exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \text{ for all } x$$

Trigonometric functions:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \text{ for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \text{ for all } x$$

Binomial series (includes the square root for $\alpha = 1/2$ and the infinite geometric series for $\alpha = -1$):

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \quad \text{for all } |x| < 1 \text{ and all complex } \alpha$$

Computers and Trigonometric Functions:Taylor series

Subject: Sine, Cosine, Tangent

Hi Dr. Math,

I was wondering, what actual functions find the values for sine, cosine, and tangent? Let's say that:

sin 48 = 0.74314482547739

cos 48 = 0.66913060635886

tan 48 = 1.1106125148292

How does a calculator or computer come up with these numbers? What does the computer actually do to the number 48 (in this case) that produces these answers? Thanks for all your help. :)

-Brian

Subject: Re: Sine, Cosine, Tangent

Hi,

Nowadays, with computer memory so readily available, the values of sine, cosine, and tangent are often stored in a table in memory.

(This memory is often located inside the same chip that carries the Central Processing Unit.) This makes it possible to produce answers very quickly.

When the computer or calculator is asked to produce a value of sine, cosine, or tangent which is not in the table, the computer exploits the fact that small changes in the angle only create small changes in the value of the function to extrapolate an answer. Because computers and calculators are only expected to give approximate answers anyway (e.g. to 10 decimal places, or what have you), this extrapolation need be done only to the necessary accuracy.

However, before there were tables, the functions had to be computed using some other method.

Consider the function sin(x). Thanks in large part to Newton, and also his student Taylor, it was discovered that when x is measured in radians:

$\sin(x) = x - x^3/3! + x^5/5! - x^7/7! + \ldots$

The equations for cosine and sine are similarly derived using Taylor series. Also, you can get cos(x) by taking the square root of 1 - sin^2(x). You can get tan(x) by dividing sin(x) by cos(x):

$\cos(x) = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots$

$\tan(x) = x + 2x^3/3! + 16x^5/5! + 272x^7/7! + 7936x^9/9! + \ldots$

The explanation for why this is true is rather long, and I hope you don't mind if I refer you to a book. You can find this in any good calculus book, for instance, by Apostol or by Courant and John.

You can think of this (infinite) sum as a series of correction factors. The first term (which is x) says: sin(x) is about equal to x. Well this is not very accurate, but if you want more accuracy, you can go to the next term: sin(x) is more like x - x^3/3!. The further you go, the more accurate you get.

It's possible to know the error from the actual value of sin(x) if you stop at a certain point in the summation. Computers and calculators need only add out to the point where the error is smaller than the desired accuracy.

One last thing: Note that for most values of x (i.e. most angles), the decimal value of sin(x) cannot be given in complete accuracy because, for most values of x, sin(x) will have a non-repeating decimal expansion.

第１回 微分とは、何故d(x^n)/dx=nx^(n-1)か。微分計算の復習。
カセットテープの回転運動。

What is dx?

What does dx mean and where does it come from?
Thanks.

From: Doctor Jeremiah
Subject: Re: dx

Hi Michael,

I assume by "dx" you mean the calculus version of dx.

Calculus is all about how to measure the slope of any arbitrary line, especially curved ones.

Consider y = 2x^2 (the x^2 means "x squared").

If you used the "normal" method to get the slope you would pick two points (lets pick (1,2) and (3,18) for this example) and then you would make a ratio of the "rise" (the difference in the y values) and the "run" (the difference in the x values)

If you did this you would have a slope of:

m = (y2-y1)/(x2-x1) = (18-2)/(3-1) = 16/2 = 8

The problem with this method is that it produces the wrong answer. The only time it's right is for a straight line. For example, pick two different points: (2,8) and (3,18)

Then you would have this slope:

m = (y2-y1)/(x2-x1) = (18-8)/(3-2) = 10/1 = 10

The curve gets flatter and flatter as the two points get closer and closer. When they get infinitely close to each other we get the most accurate answer because essentially the points are so close to each other that there is no room for any curvy bits.

If we define "dx" to be the difference between two x-values that are infinitely close to each other (an infinitely small difference in x values), and we define "dy" to be the difference between two y-values that are infinitely close to each other (an infinitely small difference in y values), then we can pick two infinitely close points and do this:

m = (y2-y1)/(x2-x1) = dy/dx

So dy/dx is the slope of a line. If we use the rules of calculus to "differentiate" our equation (using the mythical d function):

y = 2x^2

$d(y) = d(2x^2)$
$dy = 2\, d(x^2)$
$dy = 2*2x*d(x)$
$dy = 2*2x*dx$
$dy = 4x*dx$

We find that an infinitely small difference in y can be measured with this equation: $dy = 4x*dx$. But if we rearrange it slightly:

$dy = 4x*dx$
$dy/dx = 4x*dx/dx$
$dy/dx = 4x*1$
$dy/dx = 4x$

We find that the slope of $y = 2x^2$ is 4x. Notice that the slope is not a number; it actually changes depending on where in the graph we are; you can see that the slope changes by graphing $y = 2x^2$.

So the slope at any point on the graph can be found with this equation because the two points that we use to calculate with are infinitely close together (for all intents and purposes they are the same point)

And since we know the definitions of dx and dy, we could say that the slope at any point equals an infinitely small difference in y (dy) divided by an infinitely small difference in x (dx). This is absolutely true. And for a straight line graph it is the same as taking the difference of any two points.

Proof of Derivative for Function f(x) = ax^n

From: Colin
Subject: proof of the "quick form" of derivitives of A*X^N

I've been wondering if there is a proof for the "quick form" of the derivative in the ax^n case? We just learned it after using limits to calculate the derivatives. I like the quick method, but I'm the kind of person who likes to know why and how things work.

From: Doctor Peterson
Subject: Re: proof of the "quick form" of derivitives of A*X^N

Hi, Colin.
We first define derivatives using limits, then we apply that

definition to find simple rules for the derivatives of common functions, and then rarely go back to the definition again. The main value of the definition is to allow us to prove the rules and other theorems about derivatives.

Let's look at the function $f(x) = ax^n$. The derivative is

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a(x+h)^n - ax^n}{h}$$

At this point you need the binomial expansion; among other places, you can find this discussed in our FAQ on Pascal's triangle. Or see

Binomial Expansions and Pascal's Triangle
http://mathforum.org/library/drmath/view/56381.html

All that matters to us is the first two terms:

$$(a+b)^n = a^n + n\, a^{(n-1)}\, b + n(n-1)/2\, a^{(n-2)}\, b^2 + \dots$$

where the rest of the terms have whole coefficients with decreasing powers of a and increasing powers of b. Setting $a = x$ and $b = h$, and putting this into the derivative, we get

$$\lim_{h\to 0} \frac{a[x^n + nx^{(n-1)}h + n(n-1)/2\, x^{(n-2)}\, h^2 + \dots] - ax^n}{h}$$

Note that the first term of the expansion will cancel with the $-ax^n$ at the end, leaving

$$\lim_{h\to 0} \frac{anx^{(n-1)}h + an(n-1)/2\, x^{(n-2)}\, h^2 + \dots}{h}$$

Up to this point we still have the form 0/0, which means there's more to do. But now we can divide by h. The unshown terms above all have a factor of at least h^2, so we get

$$= \lim\ [anx^{(n-1)} + h(an(n-1)/2\ x^n-2 + \dots)\ ]$$

= anx^(n-1)

since the term with a factor of h goes to zero.

The main idea of limits is that when we simplify a function, as by dividing by h here, we get a continuous function that is equivalent to the original everywhere except where the latter was not defined; therefore the new function's VALUE at that point is the same as the LIMIT of the original function. In effect, we are "filling in the hole". Not all limits can be solved that easily, but when it can be done, it makes the work very easy. And now that we've done it, we don't need to bother with limits when we need to find the derivative of a polynomial.

If you have any further questions, feel free to write back.

第2回 ばねと単振動。フックの法則、運動の表現。2次の線形微分方程式。
オイラー差分近似による位置と速度の漸化式。漸化式を繰り返すことで、
積分値（単振動）が求められる。

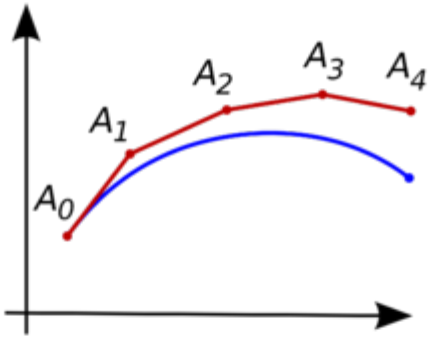
第3回 差分方程式の数値計算。運動行程式の解をオイラー法で求める。

オイラー：Euler
１７０７年４月１５日、オイラーはスイスのバーゼルに牧師の子として生まれた。父は数学が好きで ヤコブ・ベルヌーイの弟子となって勉強した。２０歳の時ペテルスブルグ王立学士院に職を得て、以後死ぬまでここから給料をもらい続けることになる。２６歳で結婚したオイラーには１３人の子供がいた。オイラーは赤ん坊を膝にのせ、子供たちと遊びながら、数学の研究論文を書いた。ニュートン力学の基本公式を初めて書き下したのはオイラーであった。変分法、剛体の力学、流体力学、音響学、航海術、船舶の設計など。月

の運動の理論〔三体問題（太陽と地球と月）〕に史上初めて計算可能な近似解を与えた。フェルマーの最終定理にも貢献。物理学者でもある。

オイラー法：Euler method　積分方法

In mathematics and computational science, the Euler method, named after Leonhard Euler, is a first-order numerical procedure for solving ordinary differential equations (ODEs) with a given initial value. It is the most basic kind of explicit method for numerical integration for ordinary differential equations.



Derivation

We want to approximate the solution of the initial value problem

dy(t)/dt　=　f(t,y(t))

by using the first two terms of the Taylor expansion of y, which represents the linear approximation around the point (t0,y(t0)) . One step of the Euler method from tn to tn+1 = tn + h is

yn+1 = yn + h・f(t,yn)

The Euler method is explicit, i.e. the solution yn + 1 is an explicit function of yi for i

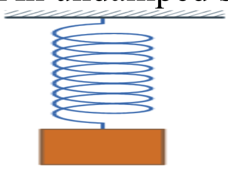
While the Euler method integrates a first order ODE, any ODE of order N can be represented as a first-order ODE in more than one variable by introducing N − 1 further variables, y', y'', ..., y(N), and formulating N first order equations in these new variables. The Euler method can be applied to the vector (y(t),y'(t),y''(t), ..., y(N)(t)　) to integrate the higher-order system.

振動：Oscillation

Oscillation is the repetitive variation, typically in time, of some measure about a central value (often a point of equilibrium) or between two or more different states. Familiar examples include a swinging pendulum and AC power.

単振動子：Simple harmonic oscillator

The simplest mechanical oscillating system is a mass attached to a linear spring subject to no other forces. Such a system may be approximated on an air table or ice surface. The system is in an equilibrium state when the spring is static. If the system is displaced from the equilibrium, there is a net restoring force on the mass, tending to bring it back to equilibrium.
The specific dynamics of this spring-mass system are described mathematically by the simple harmonic oscillator and the regular periodic motion is known as simple harmonic motion. In the spring-mass system, oscillations occur because, at the static equilibrium displacement, the mass has kinetic energy which is converted into potential energy stored in the spring at the extremes of its path.
An undamped spring–mass system is an oscillatory system.



フックの法則

In physics, simple harmonic motion (SHM) is the motion of a simple harmonic oscillator, a periodic motion that is neither driven nor damped.
A body in simple harmonic motion experiences a single force which is given by Hooke's law; that is, the force is directly proportional to the displacement x and points in the opposite direction.
Mathematically, Hooke's law states that
F=ーK・x

where
x is the displacement of the end of the spring from its equilibrium position; F is the restoring force exerted by the material; and k is the force constant (or spring constant).
xの単位はm、Fの単位はニュートンN = kg·m·s−2、k はばね定数と呼ばれる定数。個々のばね固有の値であり、ばねの強さを表している。[ニュートン毎メートル]
この法則が適用できるとき、その挙動は線型と呼ばれ、グラフに表すと正比例の直線グラフとなる。The motion is periodic: the body oscillates about an equilibrium position in a sinusoidal pattern.
フックの法則は17世紀のイギリスの物理学者、ロバート・フックが提唱したものであり、彼の名を取ってフックの法則と名づけられ

た。

運動：Dynamics of simple harmonic motion

For oscillation in a single dimension, combining Newton's second law (F = m d2x/dt2) and Hooke's law (F = −kx, as above) gives the second-order linear differential equation
F = m d2x/dt2 = −kx
where m is the mass of the body, x is its displacement from the mean position, and k is a constant. The solutions to this differential equation are sinusoidal; one solution is
$$x(t) = A\cos(\omega t + \varphi),$$
where A, ω, and φ are constants, and the equilibrium position is chosen to be the origin.[1] Each of these constants represents an important physical property of the motion: A is the amplitude, ω = 2πf is the angular frequency, and φ is the phase.

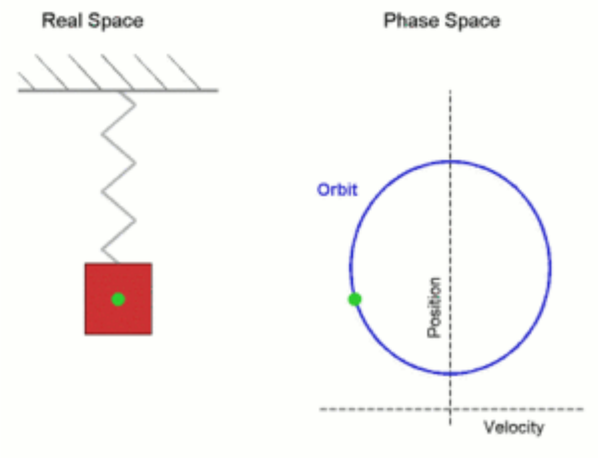
加速度と周期

Using the techniques of differential calculus, the velocity and acceleration as a function of time can be found:
$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = -A\omega\sin(\omega t + \varphi),$$
$$a(t) = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -A\omega^2\cos(\omega t + \varphi).$$
Position, velocity and acceleration of a SHM as phasorsAcceleration can also be expressed as a function of displacement.
Acceleration can also be expressed as a function of displacement:
a・x　=　-　ω^2・x

Now since ma = −mω^2x = −kx,
ω^2 = k/m.
Then since ω = 2πf,
$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}},$$
and since T = 1/f where T is the time period,
$$T = 2\pi\sqrt{\frac{m}{k}}.$$
These equations demonstrate that period and frequency are independent of the amplitude and the initial phase of the motion.

位置、速度、位相

Simple harmonic motion shown both in real space and phase space. The orbit is periodic.



単振動：free oscillationの漸化式

2次の漸化式を作ってみる。

y(t)を２階微分して見ると関数ーｙ（ｔ）になる関数を求めたい。

dˆ2y(t)/ｄｔ^2＝　-ｙ（ｔ）　を差分方程式で表す。

dy/dt＝ｚ（ｔ）　とおくと、dｚ/dt＝dˆ2y(t)/ｄｔ^2＝ーｙ（ｔ）となるので

dy/dt ＝ ｚ

dｚ/dt　＝ーｙ

差分化すると⊿y/⊿t＝（ｙt+1　-　ｙｔ）／⊿ｔ　より

yt+1　ー　ｙｔ＝yt+ｚｔ・⊿t

ｚt+1　ー　ｚｔ＝Zt ーｙｔ・⊿ｔ

行列で表すと

yt+1 ＝ 1　　⊿t　　yt

zt+1　　ー⊿t　1　　zt

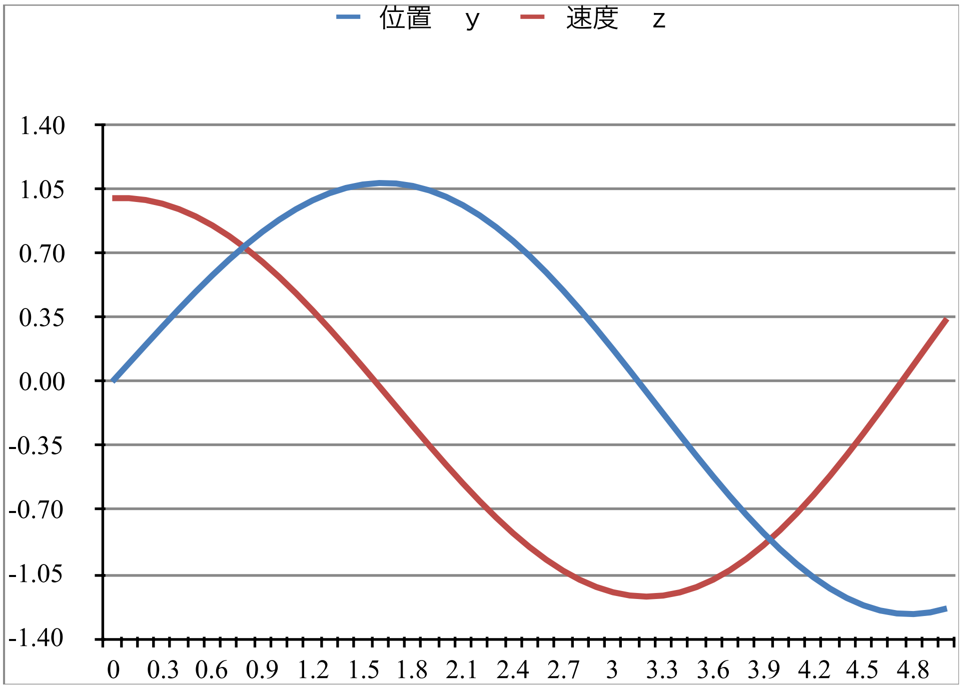
初期値 y0= 0　：位置が零　　z0= 1 ：速度が１

刻み幅 ⊿t＝ 0.1

計算結果

ｔ ｙ:位置 ｚ ：速度

0 0.00 1.00
0.1 0.10 1.00
0.2 0.20 0.99
0.3 0.30 0.97
0.4 0.40 0.94
0.5 0.49 0.90
0.6 0.58 0.85
0.7 0.67 0.79
0.8 0.74 0.73
0.9 0.82 0.65
1 0.88 0.57



答え ｙ（ｔ）＝sin（ｔ）

y'''= 一y は振動（三角関数）の微分方程式表現であった！ この解は、上の式で表される 。漸化式を使って、曲線の形を描ける。

⊿y/⊿t＝（ｙt+1 - ｙｔ）／⊿ｔ の差分はオイラー差分と呼ばれる。

オイラー差分を使うと、積分計算を行ったことになる。

角度と角速度の単位ラジアン（radian, 記号:rad）とは、国際単位系(SI)における角度（平面角）の単位である。ラジアンは、円周上でその円の半径と同じ長さの弧を切り取る2本の半径が成す角の値と定義される。1ラジアンは度数法で測ると約57.29578度に相当する。そして、180度は弧度法においては1πラジアン、360度は2πラジアンとなる。

第4回 タンクモデル。[２つのタンクの流入・流出量をモデリング]()。

Differential Equations and Flow Rate
Date: 02/14/99 at 01:05:12
From: Stephen Johnson
Subject: Differential Equations
Each of two tanks contains 100 gal of pure water. A solution containing 3 lb/gal of dye flows into Tank 1 at 5 gal/min. The well-stirred solution flows out of Tank 1 into Tank 2 at the same rate. Assuming the solution in Tank 2 is well-stirred and that this solution flows out of Tank 2 at 5 gal/min, determine the amount of dye in Tank 2 after 15 minutes.

Date: 02/14/99 at 09:47:02
From: Doctor Anthony
Subject: Re: Differential Equations

Let M1 = mass of dye in tank 1 at time t
M2 = mass of dye in tank 2 at time t

The inflow to tank 1 is 3 * 5 lbs/min = 15 lbs/min into the 100 gallon tank. The outflow is 5 * M1/100 lbs/min = .05 M1 lbs/min from the 100 gallon tank.

The differential equations are then:
dM1/dt = 15 - 0.05M1 = .05(300-M1)
dM1/(300-M1) = .05 dt and integrating
-ln(300-M1) = .05t + C
ln(300-M1) = -.05t + C

300-M1 = e^(-.05t+C) = A e^(-.05t)

and so M1 = 300 - A e^(-.05t) when t = 0, M1 = 0. Thus A = 300 and
M1 = 300[1-e^(-.05t)]

For tank 2 the differential equation is:
dM2/dt = .05M1 - .05M2
dM2/dt + .05M2 = .05M1 = 15[1-e^(-.05t)]
This is a linear equation and we multiply by the integrating factor
e^INT(.05dt) = e^(.05t):
e^(.05t) dM2/dt + .05e^(.05t) M2 = 15e^(.05t)[1-e^(-.05t)]
d[e^(.05t) M2]/dt = 15[e^(.05t) - 1] and integrating
e^(.05t) M2 = 15[(1/.05)e^(.05t) - t] + C
M2 = 15[(1/.05) - t e^(-.05t)] + C e^(-.05t)
at t = 0, M2 = 0, and so 0 = 15[1/.05] + C. So C = -300. Then
M2 = 300 - 15te^(-.05t) - 300 e^(-.05t)
M2 = 300[1-e^(-.05t)] - 15t. e^(-.05t)

When t -> infinity, M2 -> 300 which is correct.

When t = 15 this gives:
M2 = 300[1-e^(-.75)] - 15 x 15 x e^(-.75)
= 158.29 - 106.2825
= 52.0075 lbs

第5回 重力、引力。グラビティモデルと軌道、脱出速度の計算。

ふたつの質量の物体の間には、距離の２乗に反比例し、質量の積に比例するような引力が働く

Every point mass attracts every single other point mass by a force pointing along the line intersecting both points. The force is directly proportional to the product of the two masses and inversely proportional to the square of the distance between the point masses:
Fij＝G・Mi・Mj　/　Dij

where: Fij is the magnitude of the gravitational force between the two point masses, G is the gravitational constant, Mi is the mass of the first point mass, Mj is the mass of the second point mass, and Dij is the distance between the two point masses.
The gravitational attraction force between two point masses is directly proportional to the product of their masses and inversely proportional to the

square of their separation distance. The force is always attractive and acts along the line joining them.
Newton's law of universal gravitation states that every object in this universe attracts every other object with a force which is directly proportional to the product of their masses and inversely proportional to the square of distance between their centres. This is a general physical law derived from empirical observations by what Newton called induction

ニュートンの万有引力

ニュートンは、太陽を公転する地球の運動や木星の衛星の運動を統一して説明することを試み、ケプラーの法則に、運動方程式を適用することで、万有引力の法則（逆2乗の法則）を発見した。これは、『2つの物体の間には、物体の質量に比例し、2物体間の距離の2乗に反比例する引力が作用する』という法則で、力そのものは、瞬時すなわち無限大の速度で伝わると考えた。

逆２乗法則

重力に限らず、電磁波、光、電子などの物理量や強さは、一般に発生源（中心）からの距離の２乗に反比例する。

In physics, an inverse-square law is any physical law stating that some physical quantity or strength is inversely proportional to the square of the distance from the source of that physical quantity.

The inverse-square law generally applies when some force, energy, or other conserved quantity is radiated outward radially from a source.

Since the surface area of a sphere (which is $4\pi r^2$) is proportional to the square of the radius, as the emitted radiation gets farther from the source, it must spread out over an area that is proportional to the square of the distance from the source. Hence, the radiation passing through any unit area is inversely proportional to the square of the distance from the source.

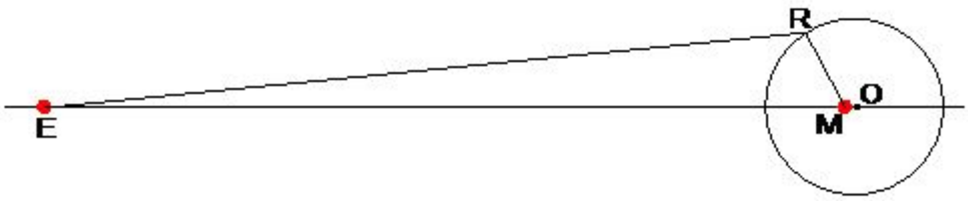
月の引力圏はアポロニウスの円

月は地球の重力圏にあるか

重力の大きさは物体の質量に比例し距離の二乗に反比例するから、2天体の場合、重力圏の大きさの比はその質量比の平方根となる。太陽と地球の場合、地球の質量比はおよそ太陽の33万分の1、重力

は距離の二乗に反比例するから、質量比の平方根に1AU（＝1億5000万km)を掛けて、地球重力圏の半径はおよそ26万kmとなる。月の軌道半径は約38万kmであるから、月は地球を中心に公転してはいるが、地球の重力圏の外にあることになる。（このため月は地球の衛星というよりは、地球と軌道を共にする惑星とも見ることができ、地球ー月の系は「連惑星系」であるとも言われる。）

月の引力圏は、ER:MR=9:1となるアポロニウスの円の内部になる



月と地球

月と地球の間の距離は38万4,400km、地球の直径は1万2,756km、月の直径は3,474km

質量はおよそ地球の0.0123倍 (1/81)。表面積（3793万km2）は地球の表面積の7.4%

地球の引力と月の引力が一致する位置は

F＝G　Me・m/ER^2＝G　M・m/MR^2　より

ER/MR=√(Me/M)=９

このことから、月の引力圏は、地球と月の距離38万4,400kmを9:1に内分する点を通るアポロニウスの円

重力加速度

重力加速度（じゅうりょくかそくど、gravitational acceleration）とは、重力の加速度のことを指す。

地球の地表付近では、どんな物体でも地面の方向への力（重力）を受けており、その大きさはその物体の質量に比例する。この比例定数を重力加速度と呼ぶ。これはその物体が自由落下する場合の加速度に一致する。

単位はメートル毎秒毎秒（m/s2）=9.80619920m/s2

ただし、質量あたりにかかる力という解釈からN/kgがより正確だとの主張もある（計算上の意味は同じである）。重力加速度は gravity の頭文字を取って g で表される。万有引力定数の G と区別するため、常に小文字で書かれる。 力加速度の値は場所によって異なるため、標準重力加速度を定めてその値を世界中で使うこととしている。

質量

重力質量（じゅうりょくしつりょう、gravitational mass）とは、質量の二つの定義のうちの一つ。ニュートンの万有引力の法則

Fij＝G・Mi・Mj / Dij

において現れる質量のM1やM2で、万有引力（重力）を起こす質量のことである。

単位は、MKS単位系では kg （キログラム）、CGS単位系では g （グラム）。

脱出速度

地球から発射される物体が、地球の引力に抗して飛び出して、人工衛星になったり、人工惑星になるために、必要なエネルギーや初速度速度を求めてみましょう。

位置エネルギー

力学的エネルギーは運動エネルギーと位置エネルギーです。 位置エネルギーは、基準の位置（例えば地球表面）からある点まで物体を動かした時消費されるエネルギーで散逸しないものが、その物体の蓄えられたとして、計算します。 質量mの物体を引力に抗してRmだけもち上げたとき、その物体の持つ位置エネルギーは地表を基準にすると、引力に抗しますのでｍｇの力でもち挙げる必要があります。そして移動距離がRですからｍｇRジュールだけ仕事をされる、物体の位置エネルギーはｍｇRです。

運動エネルギー

運動エネルギー（うんどうエネルギー、kinetic energy）は、運動し

ている物体が持つエネルギーである。言いかえれば、運動している物体を停止させるために必要なエネルギー（仕事）。 運動方程式では、質量ｘ加速度=力ですので、この両辺を積分し、距離0からrメートルまで動かす仕事（エネルギー）をして、速度が０からvになったとしましょう。この仕事量求めてみましょう。

∫m・dv/dx=∫Fdx　ただしF:一定、積分区間0=>r

左辺は　∫mv　dv　積分区間は、0->v　ですので

(1/2)mv^2=Fr

すなわち、速度v質量mの物体の運動エネルギーは、(1/2)mv^2です。

これはエネルギー積分とも呼ばれ、**「物体の運動エネルギーの変化量は、その物体に加えられた仕事量に等しい」**ことを意味する。

脱出速度の計算

M:物体A(地球)の重さ　m:物体B(ロケット)の重さ　G:万有引力定数　とします。

半径Rの位置で、位置エネルギーは　－ＧMm/R

中心からの距離rで、位置エネルギーは　－ＧMm/r

したがって、初速度V、運動中の速さをvとすると、エネルギー保存則から

(1/2)*m*(V^2)－ＧMm/R = (1/2)*m*(v^2)－ＧMm/r

全部の項にmが入っているので、約分できて、

(1/2)*(V^2)－ＧM/R = (1/2)*(v^2)－ＧM/r

mに無関係。物体B(ロケット)の質量によらず、どの大きさの質量でも、同じ結果を得ます。

ちなみに、g:地表での重力加速度　は、地球の半径をRとすれば、

g=GM/(R^2)

また、無限遠でロケットが止まらないためには、(vが0以上であるためには)、

(1/2)*(V^2) > ＧM/R　したがって、V > root(2ＧM/R) = 脱出速度　地

球で約秒速11.2km

いろいろな定数

万有引力定数：G=6.67x!0^-11 m^3/s^2/kg

地球質量：M=5.97x10^24　kg

地球半径：R=6.36^10^6

太陽質量：Ms＝1.99x10^30　kg

地球の公転半径：Re=1.50x10^11　m

衛星速度： 第一宇宙速度

地球の地表すれすれに衛星として存在するために必要な速さです。地球の重心を中心として速さvの等速円運動をした時に質量 m の物体に働く遠心力はmv^2/Rである。このとき物体に働く重力はGMm/R^2である。 遠心力と重力が釣り合うとして求める。すなわち、

mv^2/R = GMm/R^2
v=√GM/R＝7.91km/s

脱出速度：第二宇宙速度

地球の重力を振り切るために必要な最小初速度。太陽を回る人工惑星になるためには第二宇宙速度が必要である。地球の重力圏を脱出するという意味で脱出速度とも呼ばれる。 第一宇宙速度の√2倍となり、約 11.2 km/s。

地球から無限遠を基準とすると、質量 m の物体の地球表面における地球重力によって生じる位置エネルギーは、

E=-∫(-GMm/r^2)dr=-GMm/R　積分区間は無限大からRまで

これを打ち消すだけの速さの運動エネルギーmv^2/2を与える必要がある。すなわち

(1/2)mv^2=GMm/r
v=√２GM/R = 11.2km/s

Date: 6/13/96 at 9:50:25
From: dhautree
Subject: Escape velocity

How did Einstein work out how fast you need to go to get off the earth?

Date: 6/13/96 at 21:1:12
From: Doctor Luis
Subject: Re: Escape velocity

This is not really a problem Einstein worked, rather it is a well known result in Physics. It can be derived as follows:

The "escape velocity" of an object can be found by finding the work done by a gravitational field on a particle:

As you may know, the gravitational force field can be alternatively represented by the negative gradient of a special function, called the potential function

F = GMm/r^2 (m:particle's mass; M: planet's mass; r:dist from
~ center)

= - grad p (p:potential function)

Now, the work it takes for a particle with mass m to go from a pt A to a pt B is the line integral of the dot product of the force with the differential displacement around the path AB

b
W = S F * dr (the "*" means dot product)
a ~ ~

Since only the radial component of the displacement contributes to the work,

b
W = S Fr * dr , where Fr is the radial component of the force
a ~ ~

Now,
Fr = - mg (R/r)^2
~

where r is the distance of the particle from the planet, R is the planet's radius, and mg is the weigth of the particle at the surface of the planet (g is the acceleration of gravity at the planet's surface)

So,
b
W = -mgR^2 S (1/r^2) dr
a

= -mgR^2(-1/b + 1/a)
Now, setting b at infinity (this can be interpreted as: what is the energy required to move a particle from pt A out of the planet, i.e., to a point where the gravitational attraction is negligible) and setting a (the initial position) as R (the radius of the planet):

W = -mgR^2(0 + 1/R)

= -mgR

Now, the work done is equal to the change in kinetic energy (mv^2)/2
So that

(m(vf)^2)/2 - (m(vi)^2)/2 = -mgR
now, vf = 0 (initial vel.)

so that,

- m(vi)^2 = -2mgR

(vi)^2 = 2gR

or vi = sqrt(2gR)

And this is the (treshold, should I say?) velocity required for a particle to overcome the planet's gravitational field.

第6回　差分方程式。初期値と解の数列。線形性とは何か？。重ね合わせの原理。

Top > 線形差分方程式
線形差分方程式とは

線形の漸化式で表わされる方程式である。

xt=a1xt-1+a2xt-2+....+anxt-n
初期値を与えれば、数列がその解として求められる。

Xt+2=3xt+1-2xt
の場合、x1=1,x2=0からスタートすれば

{1,0,-2,-6,-14,-30,....}
の数列は、上の差分方程式から得られる。また、初期値を変えれば、下記の数列も得られる。

{1,1,1,1,1,....}
{2,4,8,16,32,....}
この数列のn番目の値xnを表わす式を求めることを、一般解を求めるという。

- 数列{xt},{yt}に対して、下記のように和とスカラー積の２つを定義することで、数列の存在する空間は線形空間の構造をもつ。線型空間（linear space）あるいはベクトル空間（vector space）とは、和とスカラー倍の定義された集合（代数系）のことである。
- {xt}+{yt}={xt+yt}
  α{xt}={αXt}
- 線形差分方程式の解全体は、 数列全体 の線形部分空間となる.

解の系列の特徴と一般解

上記の差分方程式の次の３つの解の数列x1、x２、x3はどのような性質をもつのであろうか？。

$X_1=\{1,1,1,1,1,....\}$
$X_2=\{2,4,8,16,32,....\}$
$X_3=\{1,0,-2,-6,-14,-30,....\}$
最初の解の数列はt番目の項が$1^t=１$となっており、２番目の解の数列はt番目の項が2^tとなっている。さらに、3番目の数列は、前の２つの数列の線形和の形になっている。

$X_3=2\cdot x_1+(-1/2)x_2=\{1,0,-2,-6.-14,-30,....\}$

このように、上記の差分方程式の解は、一般に

$X=c_1X_1+c_2X_2=c_1 1^t+c_2 2^t$
C_1,c_2は初期値によって異なる。

で表わされそうである。これが、差分方程式の解が線形部分空間の要素であるとの意味である。それでは、このことを一般的に成立するか調べてみよう。

シフトオペレーター：シフト作用素

複素数の無限数列の集合をVとして、その要素の$X=\{x_1,x_2,x_3,...\}$に対して、**シフト作用素E** とは

$EX=\{x_2,x_3,....\}$
を与える作用素である。この作用素は、VからVへの写像を与える作用素である。i回繰り返してシフトする作用素を　E^i　(iは整数）で表わす。E^0は、元の数列をそのまま与える。この作用素の線形結合で表わされる下記の作用素Lを、**線形差分作用素L(E)**と呼ぶ。 何故、線形作用素と呼ぶかといえば、前記の和とスカラー積の２つを定義することで、線形空間から線形空間への写像を行う作用を果たすからである。

$L(E)=c_0E^0+c_1E^1+ ....+E^m$
これは、m次差分方程式の作用素である。L(E)線形差分方程式で表わされる数列Xは、作用素表示すれば、

$L(E)x=0 <---> c_0E^0x+c_1E^1x+ ....+E^mx=0$

で表わされる。また、λの多項式　P(λ)

P(λ)=c0λ+c1λ^1+ ....+λ^m
のλに作用素Eを代入した形になっている。

- **この多項式を特性多項式と呼ぶ。**
  - 先のXt+2=3xt+1-2xtの特性多項式を示せ。
  - (答え)：P(λ)=λ^2-3λ+2
  - LX=(E^2-3E+2)(x)=E^2x-3E^1x+2E^0=0
    XはXt+2-3xt+1+2xt=0の漸化式で表わされる数を要素にもつ数列

特性多項式=0の解の性質

定理；特性多項式p(λ)=0の解をλ*とするとき、数列

X*=｛λ*,λ*^2,λ*^3,....}は、線形差分作用素で表わされるp(E)X=0

の解である。p の解が0 でなく重解を持たないならば, 差分方程式

p(E)X = 0 の解は特性多項式の解の一次結合であらわされる.

- **言い換えれば、差分方程式の一般解は次のように表わされる。**

  yt=c1λ1^t+c2λ2^t+... +cmλm^t
  そして、係数は初期条件を満たす条件からに唯一解が得られる。

重ね合わせの原理

入力utを持つ線形差分方程式を考えてみよう。

xt=a1xt-1+a2xt-2+....+anxt-n　+ut
入力系列の数列をu={u1,u2,...}とすれば、解の数列xは、線形差分作用素L(E)を用いて

L(E)x=u
で表わされる。このことから、

- **２つの入力列u,vについてL(E)x=uの解をX1、L(E)x=vの解をX2とすればL(E)x=(u+V)の解は、X1+x2で表わされる。これが重ね合わせの原理である。**

  L(E)(x1+x2)=L(E)x1+L(E)x2=u+V

- 線形差分方程式においては、入力{ut}の出力を{x1t}、入力{vt}の出力を{x2t}とする時、入力{ut+vt}の出力は、それぞれの出力の和（重ね合わせ){x1t+x2t}になる。線形系の重要な性質である。
    - この重ね合わせの原理から、　Ｌ(E)X＝u　の一般解を求めるには、その一つの特殊解に、Ｌ(E)X＝０　の一般解を加えればよいことになる。

第7回　最適化計算の事例（ニュートン法、点列を近似する曲線など）

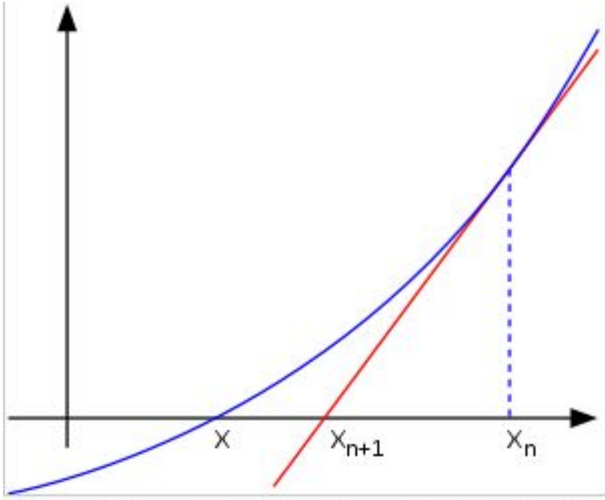
ニュートン法とは

In numerical analysis, Newton's method (also known as the Newton–Raphson method), named after Isaac Newton and Joseph Raphson, is perhaps the best known method for finding successively better approximations to the zeroes (or roots) of a real-valued function. Newton's method can often converge remarkably quickly, especially if the iteration begins "sufficiently near" the desired root. Just how near "sufficiently near" needs to be, and just how quickly "remarkably quickly" can be, depends on the problem. This is discussed in detail below. Unfortunately, when iteration begins far from the desired root, Newton's method can easily lead an unwary user astray with little warning. Thus, good implementations of the method embed it in a routine that also detects and perhaps overcomes possible convergence failures.
近似の考え方

Given a function ƒ(x) and its derivative ƒ '(x), we begin with a first guess x0 . A better approximation x1 is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

An important and somewhat surprising application is Newton–Raphson division, which can be used to quickly find the reciprocal of a number using only multiplication and subtraction.



An illustration of one iteration of Newton's method (the function f is shown in blue and the tangent line is in red). We see that x_{n+1} is a better approximation than x_n for the root x of the function f.

ニュートン法の記述

The idea of the method is as follows: one starts with an initial guess which is reasonably close to the true root, then the function is approximated by its tangent line (which can be computed using the tools of calculus), and one computes the x-intercept of this tangent line (which is easily done with elementary algebra). This x-intercept will typically be a better approximation to the function's root than the original guess, and the method can be iterated.

Suppose f : [a, b] → R is a differentiable function defined on the interval [a, b] with values in the real numbers R. The formula for converging on the root can be easily derived. Suppose we have some current approximation xn. Then we can derive the formula for a better approximation, xn+1 by referring to the diagram on the right. We know from the definition of the derivative at a given point that it is the slope of a tangent at that point.

$$f'(x_n) = \frac{\mathrm{rise}}{\mathrm{run}} = \frac{\Delta \mathrm{y}}{\Delta \mathrm{x}} = \frac{f(x_n) - 0}{x_n - x_{n+1}}.$$

Here, f ' denotes the derivative of the function f. Then by simple algebra we can derive

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

We start the process off with some arbitrary initial value x0. (The closer to the zero, the better. But, in the absence of any intuition about where the zero might lie, a "guess and check" method might narrow the possibilities to a reasonably small interval by appealing to the intermediate value theorem.)

最大値・最小値を求める方法

Newton's method can also be used to find a minimum or maximum of a function. The derivative is zero at a minimum or maximum, so minima and maxima can be found by applying Newton's method to the derivative. The iteration becomes:

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}.$$

実用上の考察

Newton's method is an extremely powerful technique -- in general the convergence is quadratic: the error is essentially squared at each step (that is, the number of accurate digits doubles in each step). However, there are some difficulties with the method.

- 1. Newton's method requires that the derivative be calculated directly. In most practical problems, the function in question may be given by a long and complicated formula, and hence an analytical expression for the derivative may not be easily obtainable. In these situations, it may be appropriate to approximate the derivative by using the slope of a line through two points on the function. In this case, the Secant method results. This has slightly slower convergence than Newton's method but does not require the existence of derivatives.
- 2 If the initial value is too far from the true zero, Newton's method may fail to converge. For this reason, Newton's method is often referred to as a local technique. Most practical implementations of Newton's method put an upper limit on the number of iterations and perhaps on the size of the iterates. If the derivative of the function is not continuous the method may fail to converge.
- 3. It is clear from the formula for Newton's method that it will fail in cases where the derivative is zero. Similarly, when the derivative is

close to zero, the tangent line is nearly horizontal and hence may "shoot" wildly past the desired root.

- 4. If the root being sought has multiplicity greater than one, the convergence rate is merely linear (errors reduced by a constant factor at each step) unless special steps are taken. When there are two or more roots that are close together then it may take many iterations before the iterates get close enough to one of them for the quadratic convergence to be apparent.

例題

Consider the problem of finding the square root of a number. There are many methods of computing square roots, and Newton's method is one.
For example, if one wishes to find the square root of 612, this is equivalent to finding the solution to
X^2＝６１２

The function to use in Newton's method is then,
f(x)= X^2-６１２

with derivative,
f '(x) =2x
With an initial guess of 10, the sequence given by Newton's method is

$$
\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} &= \quad 10 - \frac{10^2 - 612}{2 \cdot 10} &= 35.6 \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} &= 35.6 - \frac{35.6^2 - 612}{2 \cdot 35.6} &= \underline{2}6.3955056 \\
x_3 &= \quad \vdots &= \quad \vdots &= \underline{24.7}906355 \\
x_4 &= \quad \vdots &= \quad \vdots &= \underline{24.7386}883 \\
x_5 &= \quad \vdots &= \quad \vdots &= \underline{24.7386338}
\end{aligned}
$$

３次方程式　P(x) =８ｘ^3＋４ｘ^2－４ｘ－１＝０を解く

唯一の実数解の厳密解は、x=ｃｏｓ（２π/７）です。P''(x) = 48x + 8 なので少なくともx > 0ではy = P(x) は下に凸です.

一般に(an, P(an)) における接線は

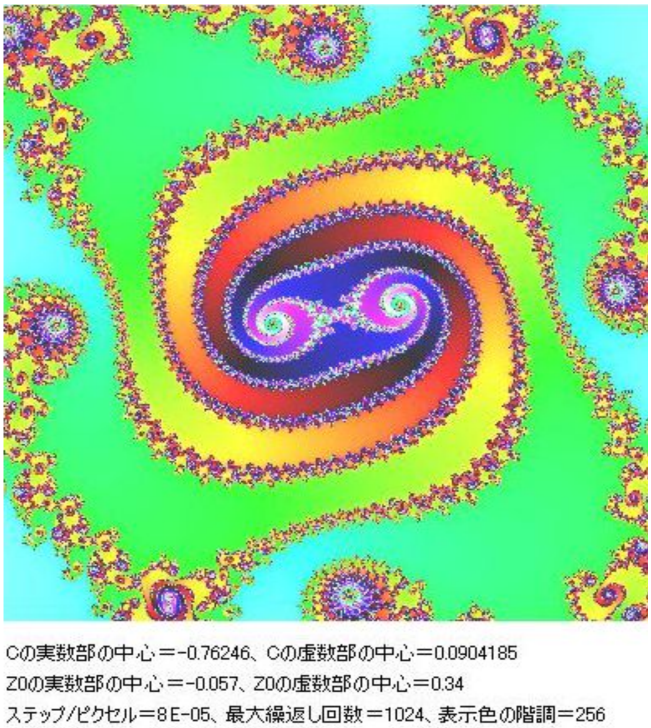
y = 4(6an^2 + 2an − 1)x − 16an^3 − 4an^2 − 1
x 軸との交点のx 座標an+1 はan+1 =16an^3 + 4an^2 + 1 {an} はこの漸化式をみたし，初項をa1 = 1とでも与えて繰り返し計算できます。一般項がきれいな式で求まるわけではありませんが，電卓を使って数

値計算するのは容易です.これが、ニュートン法です。

ジュリア集合

　複素平面上の複素変数 z に対して、前節で説明したニュートン法を使って方程式 f(z)=0 の近似値を求める場合に、数列｛zn｝が収束しないような点（初期値 z 0）が存在します。そのような点 z 0 の集合をジュリア集合といいます。

- フラクタルである図形の特徴は自己相似であること、つまりその図形の一部分が全体と相似であることです。ジュリア集合とは１９１８年にフランス人の数学者ジュリアによって発表されたものです。 当時はコンピュータが未発達であったためほとんど話題にのぼらなかったそうです。ここでジュリア集合の計算ができます
- フランスの数学者Gaston Maurice Juliaが、1918年に論文「Memoire sur l'iteration des fonctions rationnelles」で発表した。マンデルブロー集合のBenoit Mandelbrotは1924年生まれですから、マンデルブローが生まれる前からジュリア集合は知られていました。ジュリア集合とマンデルブロー集合の違いは、Z=Zˆ2+Cの繰返し計算において、マンデルブロー集合がZの初期値Z0を常にZ0=0+0iとして、C平面上に図形を描画するのに対し、ジュリア集合では、Cを任意に与え、Z0平面上に図形を描画することです。英文：Julia sets



懸垂曲線：カテナリー

カテナリー曲線　カテナリとは、電車線のことをさすが、もともとは、電車線の形づくる曲線（これを懸垂曲線という）を意味している。 In physics and geometry, the catenary is the theoretical shape a hanging chain or cable will assume when supported at its ends and acted on only by its own weight. Its surface of revolution, the catenoid, is a minimal surface and will be the shape of a soap film bounded by two circles. The curve is the graph of the hyperbolic cosine function, which has a U-like shape, similar in appearance to a parabola.

アーチ：The inverted catenary arch

The catenary is the ideal curve for an arch which supports only its own weight. When the centerline of an arch is made to follow the curve of an up-side-down (ie. inverted) catenary, the arch endures almost pure compression, in which no significant bending moment occurs inside the material. If the arch is made of individual elements (eg., stones) whose contacting surfaces are perpendicular to the curve of the arch, no significant shear forces are present at these contacting surfaces. (Shear stress is still present inside each stone, as it resists the compressive force along the shear sliding plane.) The thrust (including the weight) of the arch at its two ends is tangent to its centerline. The Gateway Arch in Saint

Louis, Missouri, United States follows the form of an inverted catenary. It is 630 feet wide at the base and 630 feet tall.

St. Louis Arch

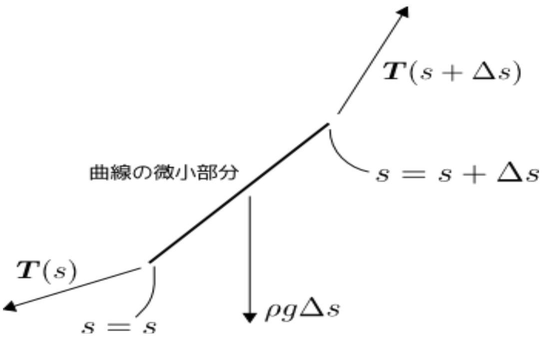
The formula for the St. Louis Arch,

y=-127.7ft cosh(x/127.7ft);757.7

is displayed inside.

懸垂線の導出：http://www.epii.jp/articles/note/math/catenary

曲線の一部を拡大するとこんな感じ:



この微小部分に働く力は、図にも示したとおり、重力 $\rho g \Delta s$と、両端に働く糸の張力 $\boldsymbol{T}(s)$, $\boldsymbol{T}(s+\Delta s)$のみです。 これに注意して力の釣り合いの式を書き下すと、

$T_{\leftrightarrow}(s+\Delta s)-T_{\leftrightarrow}(s)=0$

(1)

$T_{\updownarrow}(s+\Delta s)-T_{\updownarrow}(s)-\rho g \Delta s=0$(2)

となります。 ただし、$T_{\leftrightarrow}$ は張力 $\boldsymbol{T}$の横方向成分、 $T_{\updownarrow}$は縦方向成分をあらわし、

$\frac{T_{\updownarrow}(s)}{T_{\leftrightarrow}(s)}=\tan(\theta(s))$(3)

が成り立ちます。
まず (1) 式から張力の横方向成分 $T_{\leftrightarrow}(s)$は s に依らない定数であることが分かりますから、 その定数を、 $T :\equiv T_{\leftrightarrow}(s)$とします。
すると (3) 式の関係から、

$$T_{\updownarrow}(s) = T\tan(\theta(s))$$

(4)
とわかるので、これを (2) 式に代入することで、

$$T\tan(\theta(s+\Delta s)) - T\tan(\theta(s)) = \rho g \Delta s \quad (5)$$

を得ます。 この両辺を Δsで割ってから $\Delta s \to 0$の極限を考えることで、

$$T\frac{d}{ds}\tan(\theta(s)) = \rho g \quad ・・・・(6)$$

すなわち、

$$\frac{d}{ds}\tan(\theta(s)) = \frac{\rho g}{T} \quad (7)$$

を得ます。
さて、この微分方程式ですが、簡単に解けて、

$$\tan(\theta(s)) = \frac{\rho g}{T}s + C \quad (8)$$

です。 ここで Cは任意の定数です。 いま、s の原点を懸垂曲線の最下部にとることにすれば、その点では明らかに曲線は真横に出ている、つまり $\theta(s)|_{s=0} = 0$ なので、このときにはこの積分定数は $C = 0$となります。
従って、この境界条件のもとでは、

$$\theta(s) = \arctan\left(\frac{\rho g}{T}s\right) \quad (9)$$

となり、かなり簡単な計算で答えを求めることができました
答えとしては上ので全く問題はないのですが、θ を sの函数であらわす、という見慣れない表示ではどんな形の曲線なのか見当もつかない人がほとんどだと思います。
なので、今度は私たちのよく見慣れた、直交座標系での表示に直してみたいと思います。
方針としては、原点から距離 $s = s_0$ だけ離れた場所の座標

$(x(s_0), y(s_0))$を s_0 であらわして、その後で s_0 を消去することにより $y = f(x)$という形の、より見慣れた形に直すことにします。

まず明らかに、

(10)

が成り立ちます。

また、 $A := \frac{\rho g}{T}$として、

$$\tan(\theta(s)) = As$$

(11)

が成り立つというのが上の結果でした。 この両辺を sで微分することで、

$$\frac{1}{\cos^2(\theta(s))} \frac{d\theta}{ds} = A$$

(12)

従って (10) 式の積分は、

$$ds = \frac{1}{A} \frac{d\theta}{\cos^2 \theta}$$

(13)

と変数変換できることが分かります。

従ってまず xの方の積分は、 $\theta_0 := \arctan(As_0)$として、

(14)

さてここで、

$$\operatorname{arcsinh} x = x + \sqrt{1 + x^2}$$

(15)

であること[1]と、arcsinh が奇函数[2]であることに注意すれば、

$$x(s_0) = \frac{\operatorname{arcsinh}(As_0)}{A}$$

(16)

と計算できます。

一方、 yの方の積分はもっと簡単で、

(17)

と計算できます。

さて、あとは (16) 式と (17) 式を連立させて解くだけです。 これはとても簡単で、まず (16) 式から、

$$As_0 = \sinh(Ax)$$

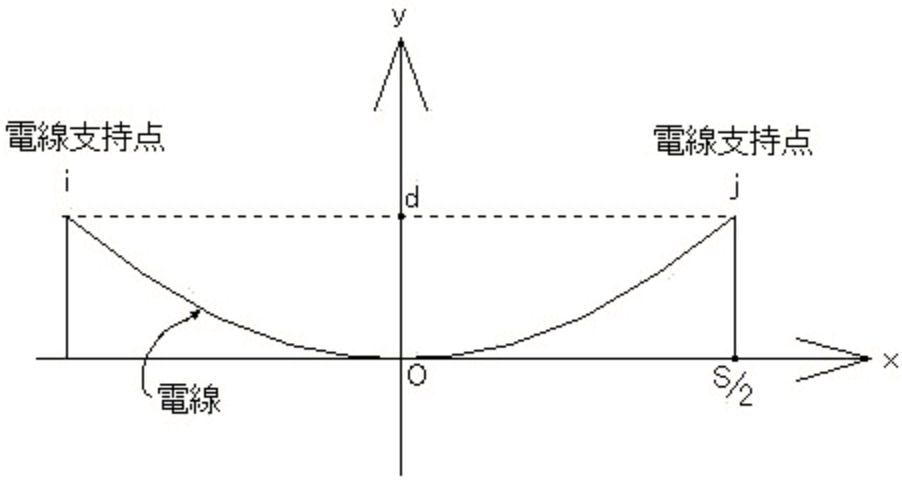
(18)
なので、これを y の式に代入して、

(19)
となります。
これは (定数倍のズレなどはあるでしょうが) 一般的に懸垂曲線の方程式と呼ばれている方程式です

電力線
横軸に径間Ｓをとり、縦軸に弛度ｄをとる。
上式をＳとｄで書き替えれば、

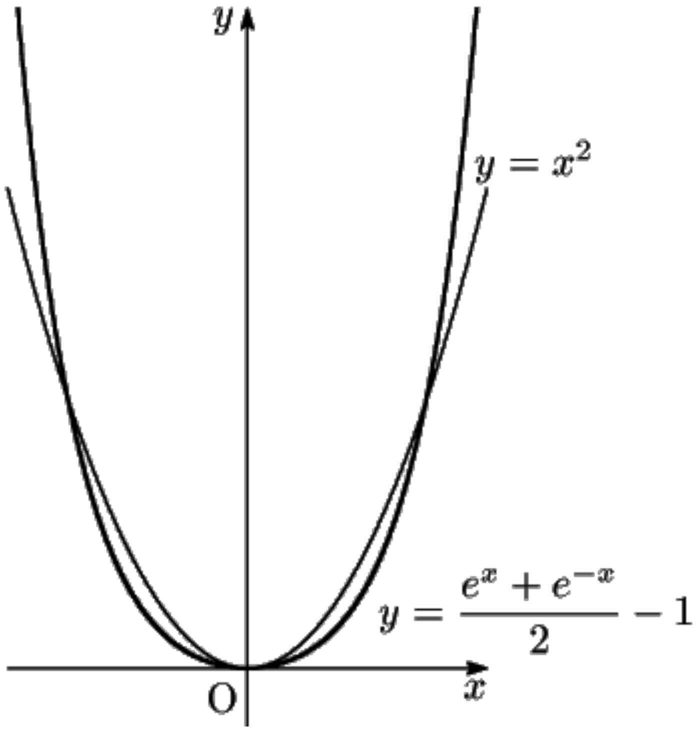
$$d = C\left(\cosh\frac{S}{2C} - 1\right)$$

ここに	d : 弛度	$y = d$
	S : 径間	$x = \frac{S}{2}$
	C : カテナリ数	$C = \frac{T}{\omega}$



カテナリ数は、長さのディメンションをもっている定数で、曲線の値を一義的に決める値である
放物線との違い
$\cosh x = \dfrac{e^x + e^{-x}}{2}$ という関数ですので，ちょっと定規とコンパスだけで作図するのは無理そうです．西洋で双曲線関数を始めて建築に取

り入れたのは，スペインのアントニ・ガウディで，19世紀末の話です．ところが，わが日本では，神社の屋根の微妙な反り返りなどがほとんど全て双曲線関数になっているのです！日本では設計図を描いてから木を削るのではなく，大工さんが鎖を垂らして曲線を決めていくので，自然と双曲線関数になってしまうのでした



二つを同じ座標平面にかいてみよう．太い方が懸垂線，細い方が放物線だ．さすがに懸垂線は指数関数でできているだけに，はじめは放物線より値が小さいが，その後はずっと大きい値をとる．

懸垂曲線：Catenaryを次の漸化式で計算する。

y(t)を２階微分して見ると元の関数ｙ（ｔ）になる関数を求めたい。

d＾2y(t)/ｄｔ^2＝ｙ（ｔ）　を差分方程式で表す。

dy/dt＝ｚ（ｔ）　とおくと、dｚ/dt＝d＾2y(t)/ｄｔ^2＝ｙ（ｔ）　となるので

dy/dt　＝　　z

dｚ/dt　　　＝　　y

差分化すると　　　⊿y/⊿t＝（ｙt+1　-　ｙｔ）／⊿ｔ　より

yt+1　ー　ｙｔ　　＝　　yt ＋ ｚｔ・⊿t
ｚt+1　ー　ｚｔ　　＝　　Zt ＋ ｙｔ・⊿ｔ

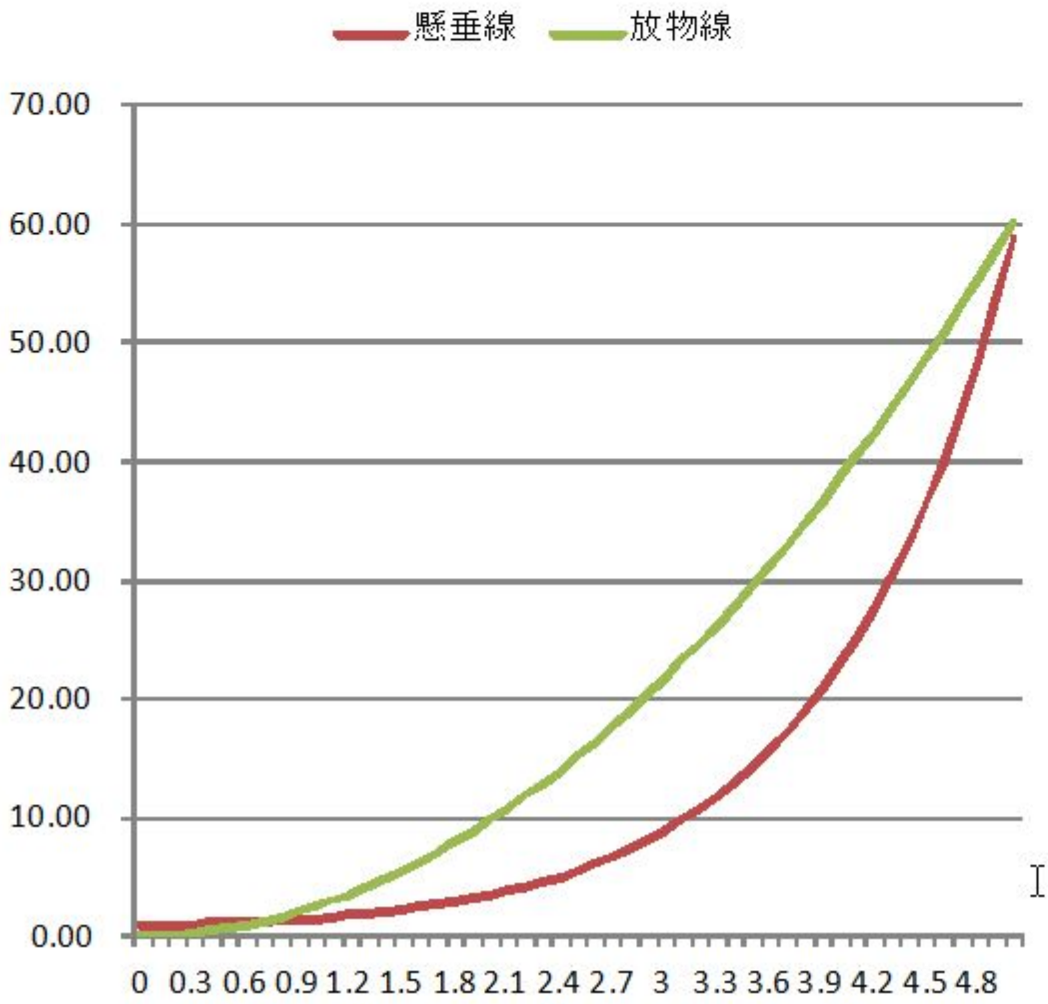
行列で表すと

yt+1 ＝	1	⊿t	yt
zt+1	⊿t	1	zt

初期値　　y0=　1
　　　　　z0=　0
刻み幅　　⊿t＝　0.1

漸化式の値　　（ｔ）　ｚ（ｔ）　t^2

0	1.00	0.00	0	
0.1	1.00	0.10	0.024	
0.2	1.01	0.20	0.096	
0.3	1.03	0.30	0.216	
0.4	1.06	0.40	0.384	
0.5	1.10	0.51	0.6	
0.6	1.15	0.62	0.864	
0.7	1.21	0.74	1.176	
0.8	1.29	0.86	1.536	答え
0.9	1.37	0.99	1.944	
1	1.47	1.12	2.4	
1.1	1.58	1.27	2.904	



答え y（t）＝｛　e^t　+　e^(-t)　｝/2

y'"=y は懸垂曲線の微分方程式表現であった！ この解は、上の式で表される漸化式を使って、曲線の形を描ける。⊿y/⊿t＝（ｙt+1　-　ｙｔ）／⊿ｔ　の差分はオイラー差分と呼ばれる。 オイラー差分を使うと、積分計算を行ったことになる。